Phase dynamics in reduced kinetic models

10^{th} Festival de Théorie

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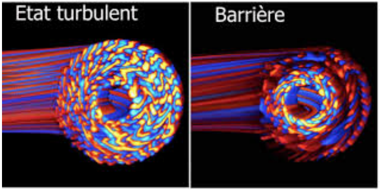
Plan

Framework

Synergy between reduced approaches and first principle physics

gyrokinetics (5D)

$$\partial_t \mathcal{F} + \{\mathcal{H}, \mathcal{F}\} = \mathcal{S} - \mathcal{C}$$



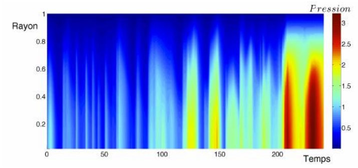
GYSELA simulation (IRFM)

- ⋆ rigorous description
- ⋆ numerical cost :

$\rho_e \longrightarrow \rho_i \longrightarrow a$ still unaccessible

Reduced models:

$$\text{Transport } \partial_t P(r,t) = \cdots$$
$$+ \text{ Zonal Flows } \partial_t \overline{\mathcal{E}}(r,t) = \cdots$$
$$+ \text{ turbulence } \partial_t \mathcal{E}(r,t) = \cdots$$



$\Rightarrow$ LH transition

- ⋆ Numerous hypothesis
- ⋆ Reasonnable cost

Reduced models can help
as long as hypothesis stay under control!

What do “reduced model” mean

Start with your preferred non-linear global equation:

$$\underbrace{\partial_t X(\mathbf{r},\mathbf{v},t)}_{time\ evolution} + \underbrace{\nabla.\left[\mathbf{u}(\mathbf{r},t)X(\mathbf{r},\mathbf{v},t)\right]}_{Flux-NonLinear} = \underbrace{S(\mathbf{r})}_{injection} - \underbrace{D(\mathbf{r},\mathbf{v},t)}_{dissipations}$$

Separate time scales: $\langle\overline{X}\rangle_t = \overline{X}$ and $\langle\widetilde{X}\rangle_t = 0$

- ⋆ Long time τ *relaxation time*

$$\partial_\tau \overline{X}(\mathbf{r},\mathbf{v},\tau) + \left\langle \nabla.\left[\widetilde{\mathbf{v}_E}(\mathbf{r},t)\widetilde{X}(\mathbf{r},\mathbf{v},t)\right]\right\rangle_t = S(\mathbf{r})$$

- ⋆ Short time t *fluctuations time*

$$\partial_t \widetilde{X} + \nabla.\left[\overline{\mathbf{u}}\,\widetilde{X} + \widetilde{\mathbf{v}_E}\,\overline{X}\right] + \nabla.\left[\widetilde{\mathbf{v}_E}\widetilde{X}\right] = -D(\mathbf{r},\mathbf{v},t)$$

How to describe the $\mathbf{E}\times\mathbf{B}$ nonlinearity $\widetilde{\mathbf{v}_E}.\nabla\widetilde{X} \propto \mathbf{b}\times\nabla\widetilde{\phi}.\nabla\widetilde{X}$??

- ⋆ Quasi-linear theory: neglect non-linear term at short times

  fast linear response + slow profile evolution
- ⋆ Shell Models: keep the fast dynamics, truncate convolution

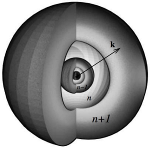
A hierarchy of reduced models

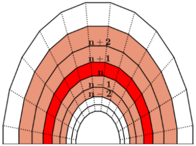
Fourier transform the $\mathbf{E} \times \mathbf{B}$ nonlinearity:

$$\mathbf{b} \times \nabla\phi \,.\, \nabla f \longrightarrow \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{b} \times \mathbf{p} \,.\, \mathbf{q}] \, \phi_{\mathbf{p}} f_{\mathbf{q}}$$

Shell models:

- ⋆ logarithmic k-grid: $k_n = k_0 \, g^n$
- ⋆ isotropy of the unknowns: $\phi_{\mathbf{p}} = \phi_p, \; f_{\mathbf{q}} = f_q$
- ⋆ only local interactions: $(p, q) = n \pm 1 \,, n \pm 2$



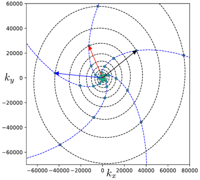


LDM approach:

- ⋆ logarithmic k-grid: $k_n = k_0 \, g^n$
- ⋆ keep the angular dependence: $\phi_{\mathbf{p}} = \phi\left(p_n, \theta_j\right) = \phi_n^j$
- ⋆ local interactions with triangle conditions

Spiral chains:

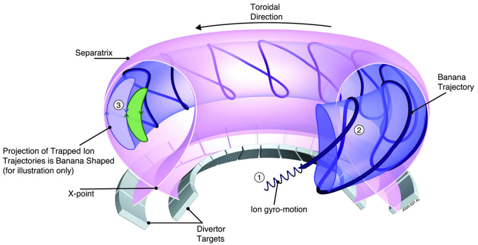
- ⋆ spiral k-grid: $\mathbf{k}_n = k_0 g^n e^{in\varphi}$
- ⋆ form exact triangles
- ⋆ relax the locality contraint $n \pm 1 \,, \; n \pm 3$



Kinetic model of trapped particles

Trapped particle motion:

- ⋆ very fast gyromotion Ω_{cs}
- ⋆ fast bounce $\omega_{bs} << \Omega_{cs}$
- ⋆ slow turbulence $\omega << \omega_{bs}$



“Bounce-average gyrokinetics“:[1]

$\alpha \approx$ toroidal angle

$\psi \approx$ radius

$$\partial_t F_s + \frac{\Omega_d E}{Z_s} \partial_\alpha F_s - \partial_\psi F_s \, \partial_\alpha \chi_s + \partial_\psi \chi_s \, \partial_\alpha F_s = 0$$

allow to describe TIM / TEM

Separate profiles (F_{0s} supposedly known) and fluctuations (δf_s) + Fourier transform:

$$\partial_t f_{sk} = i k_\alpha \partial_\psi F_{s0} \chi_{\mathbf{sk}} - i k_\alpha \frac{\Omega_d E}{Z_s} f_{\mathbf{sk}} - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{b} \times \mathbf{p} \, . \, \mathbf{q}] \, \chi_{s\mathbf{p}} \, f_{s\mathbf{q}} \, ,$$

$$C_k \phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} f_{\mathbf{sk}} \sqrt{E} dE \, .$$

[1] M. Lesur's talk right after me

Conservations

$$\partial_t f_{sk} = ik_\alpha \partial_\psi F_{s0} \chi_{s\mathbf{k}} - ik_\alpha \frac{\Omega_d E}{Z_s} f_{s\mathbf{k}} - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{b} \times \mathbf{p} \, . \, \mathbf{q}] \, \chi_{s\mathbf{p}} \, f_{s\mathbf{q}} \, ,$$

$$C_k \phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} f_{s\mathbf{k}} \sqrt{E} dE \, .$$

Entropy balance: $\mathcal{E}_{f_s} = T_{s0} \sum_{\mathbf{k}} \int_0^{+\infty} \frac{|f_{sk}|^2}{2F_{s0}} \sqrt{E} dE$

$$\partial_t \mathcal{E}_{f_s} = \left(\frac{3}{2} \kappa_{T_s} - \kappa_{ns} \right) T_{s0} \Gamma_s - \kappa_{T_s} \mathcal{Q}_s - \mathcal{D}_s$$

Electrostatic energy balance: $\mathcal{E}_\phi = \sum_{\mathbf{k}} C_k \frac{|\phi_{\mathbf{k}}|^2}{2}$

$$\partial_t \mathcal{E}_\phi = -\Omega_d \sum_s \mathcal{Q}_s - \mathcal{D}_\phi$$

Radial fluxes:

$$\Gamma_s = \sum_{\mathbf{k}} ik_\alpha \phi_{\mathbf{k}} \int_0^{+\infty} \mathcal{J}_{0s}^k f_{s\mathbf{k}}^\star \sqrt{E} dE \, ,$$

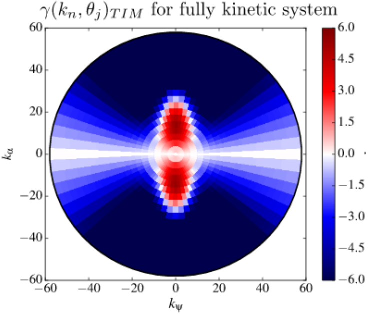
$$\mathcal{Q}_s = \sum_{\mathbf{k}} ik_\alpha \phi_{\mathbf{k}} \int_0^{+\infty} \mathcal{J}_{0s}^k f_{s\mathbf{k}}^\star E^{3/2} dE \, .$$

TIM and TEM: linear modes

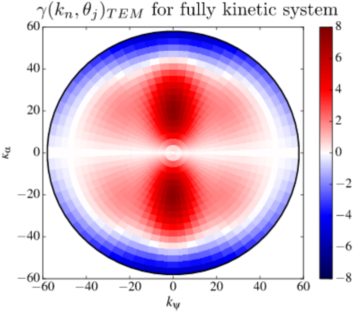
Linear dispersion relation:

$$C_k + \sum_s Z_s \int_0^{+\infty} \frac{\mathcal{J}_{0s}^2 k_\alpha \partial_\psi F_{s0}}{\omega - k_\alpha \Omega_d E / Z_s} \sqrt{E} dE = 0$$

TIM modes:



TEM modes:



- ⋆ $\gamma_{TIM} < \gamma_{TEM}$
- ⋆ TIM peak at largest scales than TEM

anisotropy is important
at least for linear injection

Plan

GOY/Sabra models

GOY[2]/Sabra[3] models:

- ⋆ logarithmic k-grid: $k_n = k_0\, g^n$
- ⋆ isotropy of the unknowns: $\phi_{\mathbf{p}} = \phi_p,\ f_{\mathbf{q}} = f_q$
- ⋆ only local interactions: $(p, q) = n \pm 1\,, n \pm 2$

Sabra version:[4]

$$\partial_t f_{sn} \quad = \quad i k_n \chi_{sn} \partial_\psi F_{s0} - i k_n \frac{\Omega_d E}{Z_s} f_{sn}^{s,j} + \alpha \frac{k_n^2}{g} \left[g^{-2} \left(\chi_{s,n-2} f_{s,n-1} - \chi_{s,n-1} f_{s,n-2} \right) - \left(\chi^\star_{s,n-1} f_{s,n+1} - \chi_{s,n+1} f^\star_{s,n-1} \right) + g^2 \left(\chi^\star_{s,n+1} f_{s,n+2} - \chi_{s,n+2} f^\star_{s,n+1} \right) \right]$$

- ⋆ source: equilibrium gradients $\qquad \partial_\psi F_{s0} \propto \dfrac{\nabla n_s}{n_s} + \left(E - \dfrac{3}{2} \right) \dfrac{\nabla T_s}{T_s}$
- ⋆ advection with precession frequency Ω_d
- ⋆ non-linear couplings: n with $\{n \pm 1; n \pm 2\}$

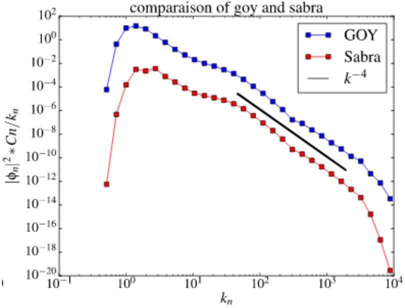
Note the complex conjugates *

GOY version: CC everywhere

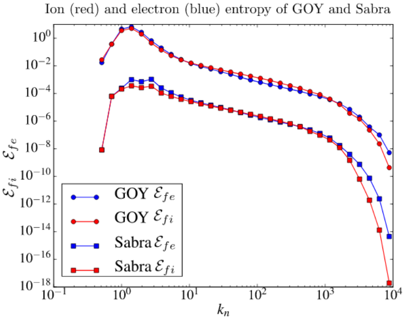
[2]Gledzer, Dokl. Akad. Nauk SSSR (1973), and Yamada, Ohkitani, J. Phys. Soc. Jpn (1987)
[3]L'vov, *et al*, PRE (1998).
[4]Shaokang Xu, PoP (2018)

GOY/Sabra: spectrae[5]

Electrostatic energy $\mathcal{E}_{\phi}$:



Entropy $\mathcal{E}_{f_s}$:



⋆ similar slopes:

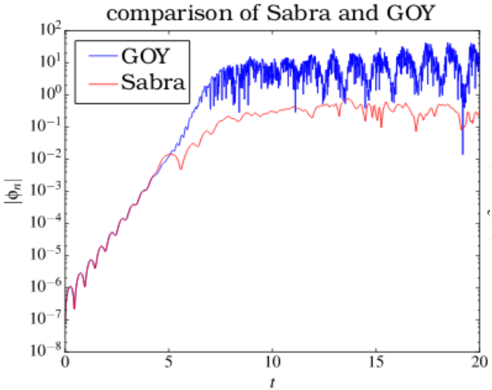
$$\mathcal{E}_{\phi} \sim k^{-4}$$
$$\mathcal{E}_{f_s} \sim k^{-1}$$

⋆ Saturation levels differ from ≈ 3 decades ?

[5] Shaokang Xu, PoP 2018, & R. Lustrat, M2 internship report

GOY/Sabra: time traces[6]

Electrostatic energy $\mathcal{E}_\phi$:



- ⋆ very different time behaviors !
- ⋆ GOY displays oscillations
- ⋆ Sabra is far more chaotic
- ⋆ only difference: phases

  GOY:
  $\partial_t f|_{NL} \propto \chi^\star_{s,n-1} f^\star_{s,n+1}$
  Sabra:
  $\partial_t f|_{NL} \propto \chi^\star_{s,n-1} f_{s,n+1}$

$\Rightarrow$ phase dependence matters!

[6] Shaokang Xu, PoP 2018

Plan

LDM equation[7]

- logarithmic k-grid: $k_n = k_0\, g^n$ + **regular** θ_k**-grid:** $\theta^{\mathbf{j}} = \mathbf{j}\,\frac{2\pi}{\mathbf{M}_\theta}$
- keep the angular dependence: $\phi_{\mathbf{p}} = \phi\left(p_n, \theta_j\right) = \phi_n^j$
- local interactions: n coupled to $n \pm 1$, $n \pm 2$

$$
\begin{aligned}
\partial_t f^{s,j}_{\ell,n} \quad = \quad & ik_\alpha \chi^s_{\ell,n} \partial_\psi F^s_{eq} - ik_\alpha \frac{E_\ell \Omega_d}{Z} f^{s,j}_{\ell,n} \\
& + \frac{k_n^2 g^{-4}}{2} \sqrt{\mu_0} \left[\chi^{\star j+r_0}_{\ell,n-2} f^{\star j-s_0}_{\ell,n-1} - \chi^{\star j-s_0}_{\ell,n-1} f^{\star j+r_0}_{\ell,n-2} + \chi^{\star j+s_0}_{\ell,n-1} f^{\star j-r_0}_{\ell,n-2} - \chi^{\star j-r_0}_{\ell,n-2} f^{\star j+s_0}_{\ell,n-1} \right] \\
& + \frac{k_n^2 g^{-2}}{2} \sqrt{\mu_0} \left[\chi^{\star j+\ell_0}_{n-1} f^{\star j-s_0}_{\ell,n+1} - \chi^{\star j-s_0}_{\ell,n+1} f^{\star j+\ell_0}_{\ell,n-1} + \chi^{\star j+s_0}_{\ell,n+1} f^{\star j-\ell_0}_{\ell,n-1} - \chi^{\star j-\ell_0}_{\ell,n-1} f^{\star j+s_0}_{\ell,n+1} \right] \\
& + \frac{k_n^2}{2} \sqrt{\mu_0} \left[\chi^{\star j+\ell_0}_{\ell,n+1} f^{\star j-r_0}_{\ell,n+2} - \chi^{\star j-r_0}_{\ell,n+2} f^{\star j+\ell_0}_{\ell,n+1} + \chi^{\star j+r_0}_{\ell,n+2} f^{\star j-\ell_0}_{\ell,n+1} - \chi^{\star j-\ell_0}_{\ell,n+1} f^{\star j+r_0}_{\ell,n+2} \right]
\end{aligned}
$$

- source: equilibrium gradients $\partial_\psi F_{s0} \propto \frac{\nabla n_s}{n_s} + \left(E - \frac{3}{2}\right) \frac{\nabla T_s}{T_s}$
- advection with precession frequency Ω_d
- non linear couplings n with $\{n \pm 1; n \pm 2\}$, **angular shifts** $\pm \mathbf{r_0}$**,** $\pm \mathbf{s_0}$**,** $\pm \ell_0$

[7] Shaokang Xu, PoP 2018

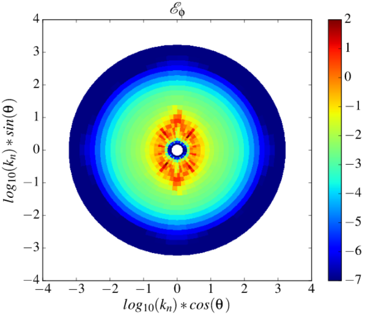
LDM simulations

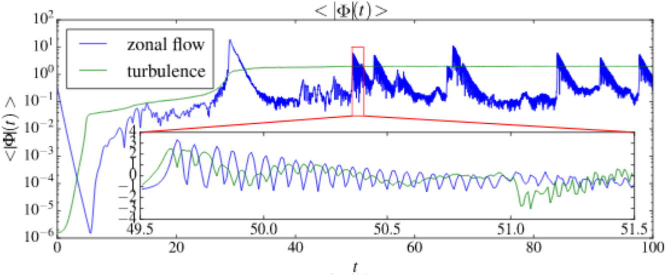
- ⋆ LDM : $\phi\left(k_n, \theta_j\right)$
- ⋆ streamers anisotropy (TEM dominate)
- ⋆ time trace $\phi(t)$

  predator (ZF) *vs* prey (turb.)

  intermittent, bursts ?

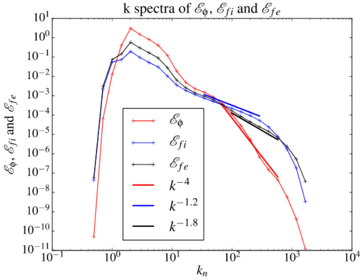
  ZF with local couplings ?

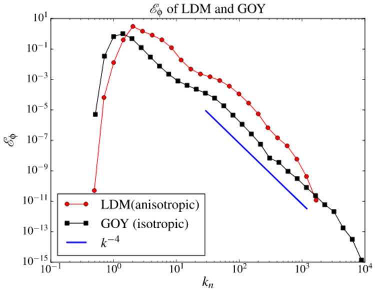




LDM simulations

k-spectrae:





- ⋆ $\mathcal{E}_\phi \sim k^{-4}$
- ⋆ $\mathcal{E}_{f_s} \sim k^{-1} - k^{-2}$
- ⋆ do not depend much on model chosen
- ⋆ do not depend much on phase dynamics

Origin of these slopes ?

Plan

Differential approximation: Motivation

- $\mathbf{E} \times \mathbf{B}$ nonlinearity:

$$\left.\frac{\partial f}{\partial t}\right|_{\mathrm{NL}} = \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{0}} [\mathbf{z} \times \mathbf{p}] \cdot \mathbf{q} \;\; \chi_{\mathbf{p}} f_{\mathbf{q}}$$

universal in plasmas

same form in 2D fluid turbulence (stream function ψ)

- Conserved quantities:

$$\mathcal{E}_f \propto f^2 \,, \quad \mathcal{E}_\phi \propto C_k \phi^2 \,.$$

- Find a model allowing to write energies spectral budgets:

$$\begin{aligned} \partial_t \mathcal{E}_k^{f_s} + \partial_k \Pi_k^{f_s} &= \mathcal{G}_k^{f_s} - \mathcal{D}_k^{f_s} \\ \partial_t \mathcal{E}_k^{\phi} + \partial_k \Pi_k^{\phi} &= \mathcal{G}_k^{\phi} - \mathcal{D}_k^{\phi} \end{aligned}$$

Π_k^f, Π_k^ϕ are spectral energy fluxes

play a key role in turbulence

approximation for $\Pi_k^{f,\phi}$?

Differential approximation: algebra

Assume:

$$k_n = k_0\, g^n = k_0\,(1+\epsilon)^n \ ,$$

with ϵ small, at fourth order:

$$\begin{aligned} k_{n+1} &= k_n + \epsilon k_n \,, \\ f_{n+1} &\approx f_n + \epsilon k_n \partial_k f_n + \frac{\epsilon^2 k_n^2}{2}\partial_k^2 f_n + \frac{\epsilon^3 k_n^3}{6}\partial_k^3 f_n + \frac{\epsilon^4 k_n^4}{24}\partial_k^4 f_n \,, \\ \dots & \quad \dots \end{aligned}$$

Plug into GOY/Sabra truncation $n \pm 1$, $n \pm 2$:

$$\begin{aligned} N &\approx \alpha \frac{k}{\chi}\,\partial_k \left[k^2 \chi^{3/2}\,\partial_k \left(k^2 \chi^{3/2} \partial_k \frac{f}{\chi} \right) \right] \\ &\approx -\alpha \frac{k}{f}\,\partial_k \left[k^2 f^{3/2}\,\partial_k \left(k^2 f^{3/2} \partial_k \frac{\chi}{f} \right) \right] \ , \end{aligned}$$

- ⋆ Antisymetric: $N[f,\chi] = -N[\chi, f]$ Poisson bracket
- ⋆ Construct energies: $N \times (\chi/k)$ or $N \times (f/k)$
- ⋆ no phase involved: χ and f are amplitudes

Passive scalar equations

$$
\begin{aligned}
\partial_t \nabla^2 \phi + \mathbf{z} \times \nabla \phi \,.\, \nabla \nabla^2 \phi &= 0,\,, \\
\partial_t n + \mathbf{z} \times \nabla \phi \,.\, \nabla n &= 0\,.
\end{aligned}
$$

Differential approximation:

$$
\begin{aligned}
\partial_t k^2 \phi &= 2\alpha \frac{k}{\phi} \partial_k \left[k^2 \phi^{3/2} \partial_k \left(k^3 \phi^{3/2} \right) \right] \,, \\
\partial_t n &= -\alpha \frac{k}{n} \partial_k \left[k^2 n^{3/2} \partial_k \left(k^2 n^{3/2} \partial_k \frac{\phi}{n} \right) \right] \,.
\end{aligned}
$$

Energy formulation: with injection $\mathcal{I}$ and dissipations $\nu_s^{\phi,n}$, $\nu_L^{\phi,n}$

$$
\begin{aligned}
\partial_t \mathcal{E}_{\phi,k} &= 2\alpha \partial_k \left[k^{5/4} \mathcal{E}_{\phi,k}^{3/4} \partial_k \left(k^{9/4} \mathcal{E}_{\phi,k}^{3/4} \right) \right] \\
&\quad + \mathcal{I}_{\phi,k} - \nu_s^{\phi} k^4 \mathcal{E}_{\phi,k}^{1/2} - \nu_L^{\phi} k^{-6} \mathcal{E}_{\phi,k} \,, \qquad (1) \\
\partial_t \mathcal{E}_{n,k} &= -\alpha \partial_k \left[\frac{\mathcal{E}_{n,k}^{3/4}}{k^{-11/4}} \partial_k \left(\frac{\mathcal{E}_{n,k}^{3/4}}{k^{-11/4}} \partial_k \frac{\mathcal{E}_{\phi,k}^{1/2}}{k \mathcal{E}_{n,k}^{1/2}} \right) \right] \\
&\quad + \mathcal{I}_{n,k} - \nu_s^{n} k^4 \mathcal{E}_{n,k} - \nu_L^{n} k^{-6} \mathcal{E}_{n,k} \,, \qquad (2)
\end{aligned}
$$

Passive scalar: spectrae

- Vorticity stationnarity:

$$\partial_k \left[k^{5/4} \mathcal{E}_{\phi,k}^{3/4} \partial_k \left(k^{9/4} \mathcal{E}_{\phi,k}^{3/4} \right) \right] \approx 0$$

$\mathcal{E}_\phi \sim k^{-3}$ or $\mathcal{E}_\phi \sim k^{-5/3}$
recover Kraichnan-Kolmogorov

- Note:

$$\partial_k \left[k^{5/4} \mathcal{E}_{\phi,k}^{3/4} \partial_k \left(k^{9/4} \mathcal{E}_{\phi,k}^{3/4} \right) \right] = \partial_k \frac{\partial_k \left(k^{9/2} \mathcal{E}_{\phi}^{3/2} \right)}{k},$$

recover the Leith model

- Passive scalar stationnarity:

$$\partial_k \left[k^{11/4} \mathcal{E}_{n,k}^{3/4} \partial_k \left(k^{11/4} \mathcal{E}_{n,k}^{3/4} \partial_k \frac{\mathcal{E}_{\phi,k}^{1/2}}{k \mathcal{E}_{n,k}^{1/2}} \right) \right] \approx 0$$

Gives six different slopes:

$$\mathcal{E}_{n,k} \sim \left\{ k^{-5}\,;\; k^{-11/3}\,;\; k^{-5/3}\,;\; k^{-1}\,;\; k^{1/3}\,;\; k^{3} \right\}$$

Signs of the fluxes

Remind the defnition of the fluxes:

$$\begin{aligned}\partial_t \mathcal{E}_\phi &= -\partial_k \Pi_k^\phi \propto \partial_k \frac{\partial_k \left(k^{9/2} \mathcal{E}_\phi^{3/2} \right)}{k}, \\ \partial_t \mathcal{E}_n &= -\partial_k \Pi_k^n \propto -\partial_k \left[k^{11/4} \mathcal{E}_n^{3/4} \partial_k \left(k^{11/4} \mathcal{E}_n^{3/4} \partial_k \sqrt{\frac{\mathcal{E}_\phi}{k^2 \mathcal{E}_n}} \right) \right]\end{aligned}$$

Constant, nonzero, fluxes correspond to :

- ⋆ $\Pi_k^\phi < 0$ for $\mathcal{E}_\phi \sim k^{-5/3}$

Inverse energy cascade

- ⋆ $\Pi_k^n > 0$ for $\mathcal{E}_\phi \sim k^{-5/3}$ and $\mathcal{E}_n \sim k^{-5/3}$
- ⋆ $\Pi_k^n > 0$ for $\mathcal{E}_\phi \sim k^{-3}$ and $\mathcal{E}_n \sim k^{-1}$

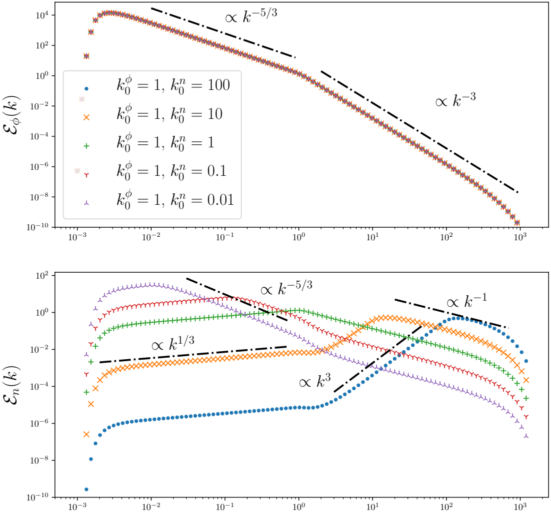
Direct enstrophy cascade
(in the case $n = k^2 \phi$)

Spectral zoo of the passive scalar

Vary injection location:

- ⋆ $\mathcal{I}_\phi$: inject vorticity at $k_0^\phi = 1.0$
- ⋆ $\mathcal{I}_n$: inject passive scalar at $k_0^n = 10^{\{-2\,;\,-1\,;\,0\,;\,1\,;\,2\}}$

$\mathcal{E}_\phi$	$\mathcal{E}_n$	Π_k^ϕ	Π_k^n
$-5/3$	$-11/3$	$-$	0
$-5/3$	$-5/3$	$-$	$+$
$-5/3$	$1/3$	$-$	0
-3	-5	0	0
-3	-1	0	$+$
-3	3	0	0



Plan

Shell model coupled to radial profiles[8]

⋆ Transport equations: density $n(r,t)$, pressure $P(r,t)$

$$\begin{aligned}\partial_t n &= \nabla\,.\,[(D_{neo} + D_{turb}\mathcal{E})\,\nabla n] + S_n(r)\\ \partial_t P &= \nabla\,.\,[(\chi_{neo} + \chi_{turb}\mathcal{E})\,\nabla P] + S_Q(r)\end{aligned}$$

⋆ Turbulence evolution: intensity $\mathcal{E}(r,t) = \sum_n \phi_n \phi_n^\star$

$$\begin{aligned}\partial_t \phi_n = \; & F_n - \nu_L k_n^{-6}\phi_n - \nu_s k_n^4 \phi_n + D_{\mathcal{E}} \nabla^2 \phi_n \\ & -\overline{\alpha}\frac{q k_n \overline{\phi}^\star}{1+k_n^2}\left[g\left(1+g^2k_n^2-q^2\right)\phi_{n+1}^\star - \left(1+\frac{k_n^2}{g^2}-q^2\right)\phi_{n-1}^\star\right] \\ & +\alpha\frac{k_n^4\left(g^2-1\right)}{1+k_n^2}\left[g^{-7}\phi_{n-2}^\star\phi_{n-1}^\star - \left(g^2+1\right)g^{-3}\phi_{n-1}^\star\phi_{n+1}^\star + g^3\phi_{n+1}^\star\phi_{n+2}^\star\right]\end{aligned}$$

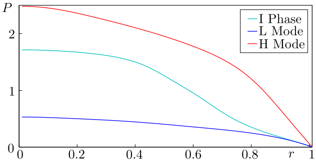
⋆ Mean flow evolution: $\overline{\phi}(r,t)$ at scale q

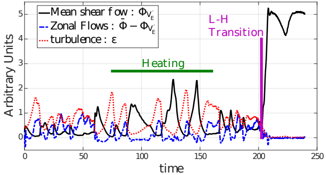
$$\partial_t\overline{\phi} = \overline{\alpha}\sum_n \frac{k_n^3 g\left(g^2-1\right)}{q}\phi_n^\star\phi_{n+1}^\star - \nu_F\left[\overline{\phi} - \phi_{V_E'}(r,t)\right]$$

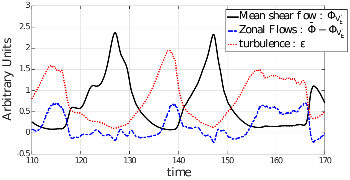
radial force balance: $\phi_{V_E'} = \frac{\eta}{q^2}\nabla P\,.\,\nabla n$

[8] V. Berionni, PoP 2017, see also Miki and Diamond for numerous works on similar system

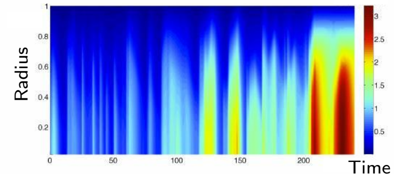
Shell model and LH transition



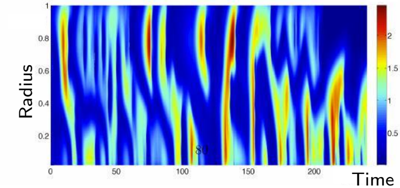




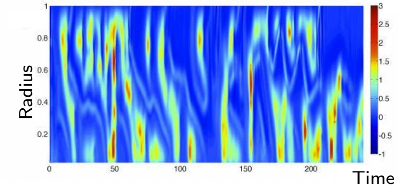
L Mode **H Mode**



$P(r,t)$



$\mathcal{E}(r,t)$



$\overline{\mathcal{E}}(r,t)$

Plan

Discussion - TODO list

- ⋆ hierarchy of reduced models available

  shell ⟶ spiral ⟶ LDM

  differential approximation

- ⋆ shell models describe LH transition

  incorrect growth rate information

  (R. Singh and R. Heinonen for similar dicussion)

  miss phase information

- ⋆ LDM model more accurate

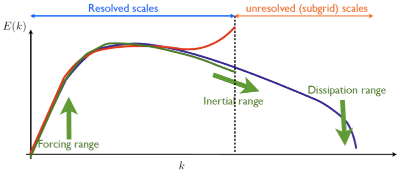
  key ingredient lacking is the nonlocal coupling

  (in k-space)

- ⋆ an UFO: differential approximation

  deserves adaptation to plasmas

- ⋆ candidates for "sub-grid" models ?



Plan

Back to TEM+TIM system : phase representation

We have to solve :

$$\partial_t f_{s\mathbf{k}} = ik_\alpha \partial_\psi F_{s0} \mathcal{J}_{0s} \phi_{\mathbf{k}} - ik_\alpha \frac{\Omega_d E}{Z_s} f_{s,\mathbf{k}} - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z} \times \mathbf{p} \, . \, \mathbf{q}] \, \mathcal{J}_{0s} \phi^\star_{\mathbf{p}} \, f^\star_{s\mathbf{q}} \, ,$$

$$C_{\mathbf{k}} \phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} f_{s\mathbf{k}} \sqrt{E} dE \, .$$

Decompose the distribution function into phase/amplitude: $f_{s\mathbf{k}} = |f_{s\mathbf{k}}| \, e^{i\varphi_{\mathbf{k}}}$

$$\partial_t |f_{s\mathbf{k}}| = -k_\alpha \partial_\psi F_{s0} \, \Im \left[\mathcal{J}_{0s} \phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}} \right] - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}} [\mathbf{z} \times \mathbf{p} \, . \, \mathbf{q}] \, \Re \left[\mathcal{J}_{0s} \phi^\star_{\mathbf{p}} \, |f_{s\mathbf{q}}| \, e^{-i\left(\varphi_{\mathbf{q}} + \varphi_{\mathbf{k}}\right)} \right] \, ,$$

$$\partial_t \varphi_{f_s\mathbf{k}} = k_\alpha \partial_\psi F_{s0} \frac{\Re \left[\mathcal{J}_{0s} \phi_{\mathbf{k}} \, e^{-i\varphi_{\mathbf{k}}} \right]}{|f_{s\mathbf{k}}|} - k_\alpha \frac{\Omega_d E}{Z_s}$$
$$- \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z} \times \mathbf{p} \, . \, \mathbf{q}] \, \Im \left[\mathcal{J}_{0s} \phi^\star_{\mathbf{p}} \, e^{-i\left(\varphi_{\mathbf{q}} + \varphi_{\mathbf{k}}\right)} \right] \frac{|f_{s\mathbf{q}}|}{|f_{s\mathbf{k}}|} \, ,$$

$$C_{\mathbf{k}} \phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} \, |f_{s\mathbf{k}}| \, e^{i\varphi_{\mathbf{k}}} \, .$$

Phase representation of TEM+TIM

$$\partial_t |f_{s\mathbf{k}}| = -k_\alpha \partial_\psi F_{s0} \Im \left[\mathcal{J}_{0s} \phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}} \right] - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}} [\mathbf{z} \times \mathbf{p} \,.\, \mathbf{q}] \Re \left[\mathcal{J}_{0s} \phi^\star_{\mathbf{p}} |f_{s\mathbf{q}}| \, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)} \right] ,$$

$$\partial_t \varphi_{\mathbf{k}} = -k_\alpha \frac{\Omega_d E}{Z_s} + k_\alpha \partial_\psi F_{s0} \frac{\Re \left[\mathcal{J}_{0s} \phi_{\mathbf{k}} \, e^{-i\varphi_{\mathbf{k}}} \right]}{|f_{s\mathbf{k}}|} - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z} \times \mathbf{p} \,.\, \mathbf{q}] \Im \left[\mathcal{J}_{0s} \phi^\star_{\mathbf{p}} \, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)} \right] \frac{|f_{s\mathbf{q}}|}{|f_{s\mathbf{k}}|} ,$$

$$C_{\mathbf{k}} \phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} \, |f_{s\mathbf{k}}| \, e^{i\varphi_{\mathbf{k}}} \left(= C_k \, |\phi_k| \, e^{i\overline{\varphi_{\mathbf{k}}}} \right) .$$

- Injection wrt amplitude: linear, due to equilibrium gradients $\propto \partial_\psi F_{s0}$
  contains the radial fluxes of heat and particles
  anisotropy: $k_\alpha = k \sin \theta_k$
  phase relationship: $\Im \left[J_{0s} \phi_{\mathbf{k}} e^{-i\varphi_k} \right] \propto \sin \left(\overline{\varphi_{\mathbf{k}}} - \varphi_{\mathbf{k}} \right)$
  $\Rightarrow$ null for $\overline{\varphi_{\mathbf{k}}} - \varphi_{\mathbf{k}} = 0$
  correspond to **stable drift waves**

Phase representation of TEM+TIM

$$\partial_t |f_{s\mathbf{k}}| = -k_\alpha \partial_\psi F_{s0} \Im\left[\mathcal{J}_{0s}\phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}}\right] - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}} [\mathbf{z}\times\mathbf{p}\,.\,\mathbf{q}]\, \Re\left[\mathcal{J}_{0s}\phi^\star_{\mathbf{p}} |f_{s\mathbf{q}}|\, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)}\right],$$

$$\partial_t \varphi_{\mathbf{k}} = -k_\alpha \frac{\Omega_d E}{Z_s} + k_\alpha \partial_\psi F_{s0} \frac{\Re\left[\mathcal{J}_{0s}\phi_{\mathbf{k}}\, e^{-i\varphi_{\mathbf{k}}}\right]}{|f_{s\mathbf{k}}|} - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z}\times\mathbf{p}\,.\,\mathbf{q}]\, \Im\left[\mathcal{J}_{0s}\phi^\star_{\mathbf{p}}\, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)}\right] \frac{|f_{s\mathbf{q}}|}{|f_{s\mathbf{k}}|},$$

$$C_{\mathbf{k}}\phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} |f_{s\mathbf{k}}|\, e^{i\varphi_{\mathbf{k}}} \left(= C_k |\phi_k|\, e^{i\overline{\varphi_{\mathbf{k}}}}\right).$$

- ⋆ Injection wrt amplitude
- ⋆ Nonlinear transfers: triangles $\mathbf{k}+\mathbf{p}+\mathbf{q}=0$

phase relationship: $\Re\left[\mathcal{J}_{0s}\phi^\star_{\mathbf{p}} |f_{s\mathbf{q}}|\, e^{-i\left(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}}\right)}\right] \propto \cos\left(\varphi_{\mathbf{k}}+\varphi_{\mathbf{q}}+\overline{\varphi_{\mathbf{p}}}\right)$

max for $\varphi_{\mathbf{k}}+\overline{\varphi_{\mathbf{p}}}+\varphi_{\mathbf{q}}=0$

Phase representation of TEM+TIM

$$\partial_t |f_{s\mathbf{k}}| = -k_\alpha \partial_\psi F_{s0} \Im\left[\mathcal{J}_{0s}\phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}}\right] - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}} [\mathbf{z}\times\mathbf{p}\,.\,\mathbf{q}]\,\Re\left[\mathcal{J}_{0s}\phi^\star_{\mathbf{p}}\,|f_{s\mathbf{q}}|\,e^{-i(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}})}\right],$$

$$\partial_t\varphi_{\mathbf{k}} = -k_\alpha \frac{\Omega_d E}{Z_s} + k_\alpha \partial_\psi F_{s0} \frac{\Re\left[\mathcal{J}_{0s}\phi_{\mathbf{k}}\, e^{-i\varphi_{\mathbf{k}}}\right]}{|f_{s\mathbf{k}}|} - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z}\times\mathbf{p}\,.\,\mathbf{q}]\,\Im\left[\mathcal{J}_{0s}\phi^\star_{\mathbf{p}}\, e^{-i(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}})}\right] \frac{|f_{s\mathbf{q}}|}{|f_{s\mathbf{k}}|},$$

$$C_{\mathbf{k}}\phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s}\,|f_{s\mathbf{k}}|\; e^{i\varphi_{\mathbf{k}}} \left(= C_k\,|\phi_k|\,e^{i\overline{\varphi_{\mathbf{k}}}}\right).$$

- Injection wrt amplitude
- Nonlinear transfers
- ballistic phase: $\propto Z_s$ linear growth/decrease of phase $\propto k_\alpha, E$

Phase representation of TEM+TIM

$$\partial_t |f_{s\mathbf{k}}| = -k_\alpha \partial_\psi F_{s0} \Im \left[\mathcal{J}_{0s} \phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}} \right] - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}} [\mathbf{z} \times \mathbf{p} \,.\, \mathbf{q}] \Re \left[\mathcal{J}_{0s} \phi^\star_{\mathbf{p}} |f_{s\mathbf{q}}| e^{-i(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}})} \right],$$

$$\partial_t \varphi_{\mathbf{k}} = -k_\alpha \frac{\Omega_d E}{Z_s} + k_\alpha \partial_\psi F_{s0} \frac{\Re \left[\mathcal{J}_{0s} \phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}} \right]}{|f_{s\mathbf{k}}|} - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z} \times \mathbf{p} \,.\, \mathbf{q}] \Im \left[\mathcal{J}_{0s} \phi^\star_{\mathbf{p}} e^{-i(\varphi_{\mathbf{q}}+\varphi_{\mathbf{k}})} \right] \frac{|f_{s\mathbf{q}}|}{|f_{s\mathbf{k}}|},$$

$$C_{\mathbf{k}} \phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} |f_{s\mathbf{k}}| e^{i\varphi_{\mathbf{k}}} \left(= C_k |\phi_k| e^{i\overline{\varphi_{\mathbf{k}}}} \right).$$

- ⋆ Injection wrt amplitude
- ⋆ Nonlinear transfers
- ⋆ ballistic phase
- ⋆ phase coupling wrt energy E:

background gradients $\propto \partial_\psi F_{s0}$

anisotropy: $\propto k_\alpha = k \sin\theta_k$

phase relationship: $\Re \left[J_{0s} \phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}} \right] \propto \cos\left(\overline{\varphi_{\mathbf{k}}} - \varphi_{\mathbf{k}} \right)$

$\Rightarrow$ max for $\overline{\varphi_{\mathbf{k}}} - \varphi_{\mathbf{k}} = 0$

Kuramoto model: $\boxed{d_t\theta_i = \omega_i + Kr\sin(\psi - \theta_i)}$

Phase representation of TEM+TIM

$$\partial_t |f_{s\mathbf{k}}| = -k_\alpha \partial_\psi F_{s0} \, \Im \left[\mathcal{J}_{0s} \phi_{\mathbf{k}} e^{-i\varphi_{\mathbf{k}}} \right] - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}} [\mathbf{z} \times \mathbf{p} \, . \, \mathbf{q}] \, \Re \left[\mathcal{J}_{0s} \phi^\star_{\mathbf{p}} \, |f_{s\mathbf{q}}| \, e^{-i(\varphi_{\mathbf{q}} + \varphi_{\mathbf{k}})} \right],$$

$$\partial_t \varphi_{\mathbf{k}} = -k_\alpha \frac{\Omega_d E}{Z_s} + k_\alpha \partial_\psi F_{s0} \frac{\Re \left[\mathcal{J}_{0s} \phi_{\mathbf{k}} \, e^{-i\varphi_{\mathbf{k}}} \right]}{|f_{s\mathbf{k}}|} - \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} [\mathbf{z} \times \mathbf{p} \, . \, \mathbf{q}] \, \Im \left[\mathcal{J}_{0s} \phi^\star_{\mathbf{p}} \, e^{-i(\varphi_{\mathbf{q}} + \varphi_{\mathbf{k}})} \right] \frac{|f_{s\mathbf{q}}|}{|f_{s\mathbf{k}}|},$$

$$C_{\mathbf{k}} \phi_{\mathbf{k}} = \sum_s Z_s \int_0^{+\infty} \mathcal{J}_{0s} \, |f_{s\mathbf{k}}| \, e^{i\varphi_{\mathbf{k}}} \left(= C_k \, |\phi_k| \, e^{i\overline{\varphi_{\mathbf{k}}}} \right).$$

- ⋆ Injection wrt amplitude
- ⋆ Nonlinear transfers
- ⋆ ballistic phase
- ⋆ phase coupling wrt energy E
- ⋆ phase coupling wrt k:

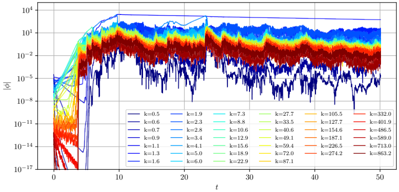
triangles $\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$

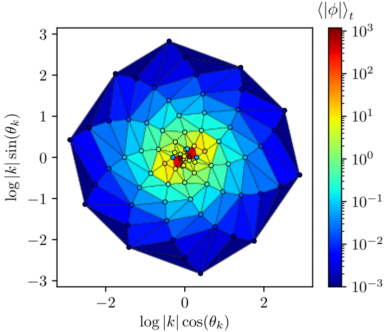
$$\Im \left[\mathcal{J}_{0s} \phi^\star_{\mathbf{p}} e^{-i(\varphi_{\mathbf{q}} + \varphi_{\mathbf{k}})} \right] \propto \sin \left(\varphi_{\mathbf{k}} + \overline{\varphi_{\mathbf{p}}} + \varphi_{\mathbf{q}} \right)$$

$\Rightarrow$ minimal for maximal nonlinear transfers

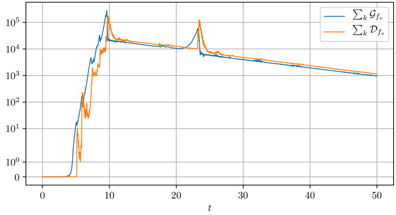
Phase dynamics in TEM

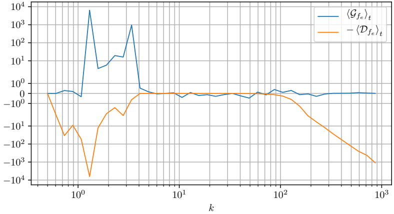
TEM driven simulation + phase representation



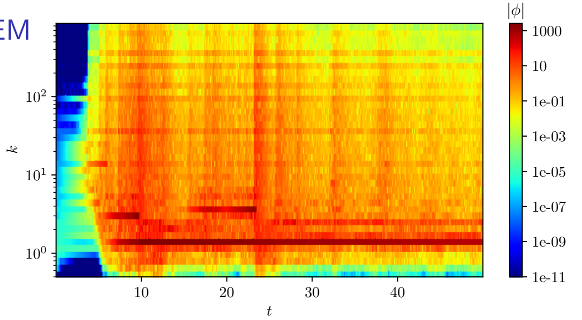


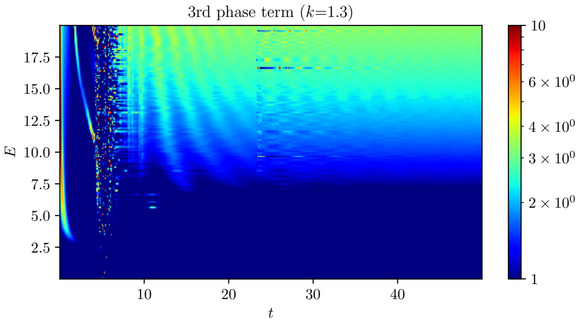
Phase dynamics in TEM



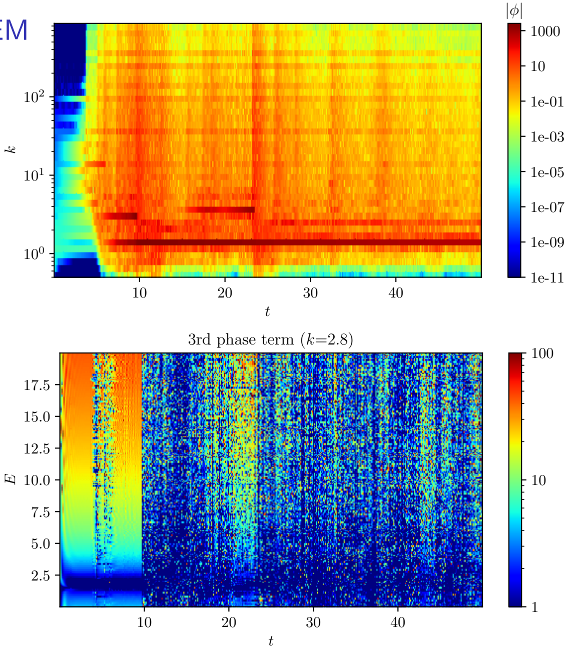


Phase dynamics in TEM





Phase dynamics in TEM



Phase dynamics in TEM

Perspectives

- ⋆ scalaire passif :

solution analytique zones dissipatives ?

$Pr << 1$: $\mathcal{E}_n$ suramorti avec $\mathcal{E}_\phi \propto k^{-3}$

$$d_k \left[k^{11/4} \mathcal{E}_n^{3/4} \partial_k \left(k^{11/4} \mathcal{E}_n^{3/4} d_k \sqrt{\frac{k^{-3}}{k^2 \mathcal{E}_n}} \right) \right] + \nu_s^n k^2 \mathcal{E}_n = 0$$

$Pr >> 1$: $\mathcal{E}_\phi$ suramorti

$$d_k \left[\frac{d_k \left(k^{17/2} \mathcal{E}_\phi^{3/2} \right)}{k} \right] - \nu_s^\phi k^4 \mathcal{E}_\phi = 0$$

- ⋆ anisotropie : en tenir compte en posant :

$$\begin{aligned} p &= k\left(1+\epsilon_p\right) \\ q &= k\left(1+\epsilon_q\right) \end{aligned}$$

$\Rightarrow$ développement limité à deux variables $\left(\epsilon_p\ ,\ \epsilon_q\right)$

$\Rightarrow$ déformation des angles $\alpha_p = \arccos \frac{p^2+k^2-q^2}{2pk}$

$$= \arccos \frac{1+\epsilon_p\left(2+\epsilon_p\right)-\epsilon_q\left(2+\epsilon_q\right)}{2+2\epsilon_p}$$

- ⋆ cas gyrocinétique : gyromoyennes ??