Mean Responses to Symmetry Breaking Perturbations in Disordered Systems

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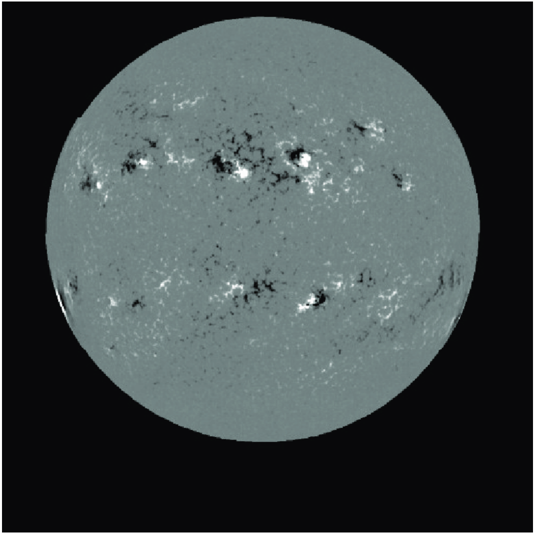
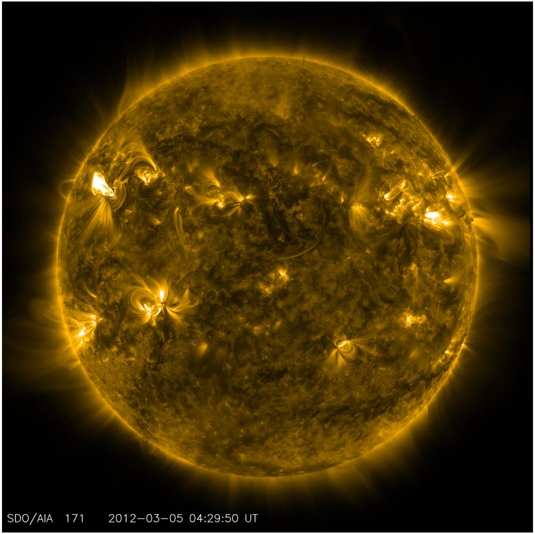
Collaboration with Michael Proctor (Cambridge), Joanne Mason (Exeter) and Alastair Rucklidge (Leeds).

Builds on previous work with Alice Courvoisier (Leeds).

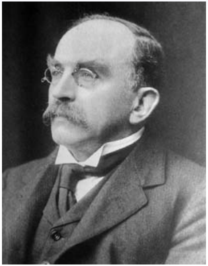
The Sun's magnetic field



Solar magnetic field viewed in the corona



Solar dynamo theory



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How could a rotating body such as the Sun become a magnet?

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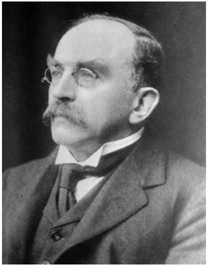

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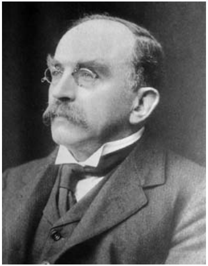

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How could a rotating body such as the Sun become a magnet?
Answer: dynamo action.

However, what is still an unanswered question:

How could a rotating body such as the Sun become a large-scale magnet?

Mean field models of MHD turbulence

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But does this produce sensible answers?

Formal calculation of mean field coefficients

Starting point is a forced symmetric MHD state with small-scale velocity $\boldsymbol{U}(\boldsymbol{x}, t)$ and small-scale magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$. This could be the result of a small-scale dynamo or via forcing in the induction equation.

The governing equations are those of incompressible MHD:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\nabla P + \boldsymbol{B} \cdot \nabla \boldsymbol{B} + Re^{-1} \nabla^2 \boldsymbol{U} + \boldsymbol{F},$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{U} + Rm^{-1} \nabla^2 \boldsymbol{B},$$

$$\nabla \cdot \boldsymbol{U} = 0, \qquad \nabla \cdot \boldsymbol{B} = 0.$$

Here $\boldsymbol{F}$ is some forcing function, Re and Rm are the fluid and magnetic Reynolds numbers.

We assume that $\boldsymbol{F}$, Re and Rm are such that there is no mean field or flow.

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Here $\boldsymbol{F}$ is some forcing function, Re and Rm are the fluid and magnetic Reynolds numbers.
We assume that $\boldsymbol{F}$, Re and Rm are such that there is no mean field or flow.
Now suppose that the system is perturbed by a small perturbation $\boldsymbol{u}$, $\boldsymbol{b}$. The linearised perturbation equations are:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{U} = -\nabla p + \boldsymbol{B} \cdot \nabla \boldsymbol{b} + \boldsymbol{b} \cdot \nabla \boldsymbol{B} + Re^{-1} \nabla^2 \boldsymbol{u},$$

$$\frac{\partial \boldsymbol{b}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{b} + \boldsymbol{u} \cdot \nabla \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{U} + Rm^{-1} \nabla^2 \boldsymbol{b},$$

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \nabla \cdot \boldsymbol{b} = 0.$$

Formal calculation of mean field coefficients

Decompose the disturbance flow and field into mean and fluctuating components.
Write $\boldsymbol{u} = \langle \boldsymbol{u} \rangle + \boldsymbol{u}'$, etc., where $\langle\ \rangle$ denotes an average over intermediate spatial scales.

The mean equations are:

$$\frac{\partial \langle \boldsymbol{u} \rangle}{\partial t} + \frac{\partial}{\partial x_j} \langle U_j \boldsymbol{u}' + u_j' \boldsymbol{U} \rangle = -\nabla \langle P \rangle + \frac{\partial}{\partial x_j} \langle B_j \boldsymbol{b}' + b_j' \boldsymbol{B} \rangle + Re^{-1} \nabla^2 \langle \boldsymbol{u} \rangle,$$

$$\frac{\partial \langle \boldsymbol{b} \rangle}{\partial t} + \langle \boldsymbol{U} \cdot \nabla \boldsymbol{b}' \rangle + \langle \boldsymbol{u}' \cdot \nabla \boldsymbol{B} \rangle = \langle \boldsymbol{B} \cdot \nabla \boldsymbol{u}' \rangle + \langle \boldsymbol{b}' \cdot \nabla \boldsymbol{U} \rangle + Rm^{-1} \nabla^2 \langle \boldsymbol{b} \rangle.$$

And the fluctuating components (assuming $\langle \boldsymbol{u} \rangle$ and $\langle \boldsymbol{b} \rangle$ are uniform to leading order) are described by:

$$\begin{aligned}\frac{\partial \boldsymbol{u}'}{\partial t} + \left(\boldsymbol{U} \cdot \nabla \boldsymbol{u}' + \boldsymbol{u}' \cdot \nabla \boldsymbol{U}\right)' + \langle \boldsymbol{u} \rangle \cdot \nabla \boldsymbol{U} = -\nabla P' + \left(\boldsymbol{B} \cdot \nabla \boldsymbol{b}' + \boldsymbol{b}' \cdot \nabla \boldsymbol{B}\right)' \\ + \langle \boldsymbol{b} \rangle \cdot \nabla \boldsymbol{B} + Re^{-1} \nabla^2 \boldsymbol{u}',\end{aligned}$$

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We are interested in the evolution of the mean fields $\langle \mathbf{u} \rangle$, $\langle \boldsymbol{b} \rangle$. If we could solve explicitly for the fluctuating fields, in terms of the mean fields, then we would have a closed system.
But typically this is not the case.

Mean field ansatz

However, the linearity of the fluctuating equations in the mean fields allows us to write the mean field equations in the form

$$\frac{\partial \langle u_i \rangle}{\partial t} + \frac{\partial}{\partial x_j} \left(\Gamma^U_{ijl} \langle u_l \rangle + \Gamma^B_{ijl} \langle b_l \rangle \right) = -\frac{\partial}{\partial x_i} \langle p \rangle + Re^{-1} \nabla^2 \langle u_i \rangle,$$

$$\frac{\partial \langle b_i \rangle}{\partial t} = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\alpha^B_{kl} \langle b_l \rangle + \alpha^U_{kl} \langle u_l \rangle \right) + Rm^{-1} \nabla^2 \langle b_i \rangle.$$

Hence the evolution of the mean flow and magnetic field is governed by four tensors (or pseudo-tensors): $\mathbf{\Gamma}^U$, $\mathbf{\Gamma}^B$, $\boldsymbol{\alpha}^U$, $\boldsymbol{\alpha}^B$. These tensors depend on the statistics of the basic state flow and field $(\boldsymbol{U}, \boldsymbol{B})$.

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Two cases where progress can be made:

(i) Small *Rm*. No small-scale dynamo, so need forcing in induction equation (Courvoisier, H + Proctor 2010 *Astron. Nachr.*).

(ii) Short sudden turbulence (Small Strouhal number).

Short sudden approximation

Perturbations determined by:

$$\boldsymbol{u}' = \tau(\langle \boldsymbol{b} \rangle \cdot \nabla \boldsymbol{B} - \langle \boldsymbol{u} \rangle \cdot \nabla \boldsymbol{U}), \quad \boldsymbol{b}' = \tau(\langle \boldsymbol{b} \rangle \cdot \nabla \boldsymbol{U} - \langle \boldsymbol{u} \rangle \cdot \nabla \boldsymbol{B}).$$

First it should be noted that $\boldsymbol{\Gamma}^U$ vanishes under this approximation. Substituting for $\boldsymbol{u}'$ and $\boldsymbol{b}'$ gives

$$\begin{aligned} \alpha^B_{il} &= \tau \epsilon_{ipq} \left(U_p \frac{\partial U_q}{\partial x_l} - B_p \frac{\partial B_q}{\partial x_l} \right), \\ \alpha^U_{il} &= -2\tau \epsilon_{imn} \gamma_{mnl}, \\ \Gamma^B_{ijl} &= 2\tau(\gamma_{ijl} + \gamma_{jil}), \end{aligned}$$

where

$$\gamma_{ijl} = \left\langle U_i \frac{\partial B_j}{\partial X_l} \right\rangle.$$

The expression for $\boldsymbol{\alpha}^{(1)}$, which first appeared in Pouquet, Frisch & Léorat (1976), extends the classical α-effect from hydrodynamic to MHD basic states.

Since $\boldsymbol{\Gamma}^B$ is symmetric in its first two arguments, it will vanish if the statistics of the basic state are isotropic.

Coefficients of $\boldsymbol{\gamma}$ are *a priori* almost unconstrained, though since $\langle \boldsymbol{u} \rangle$, $\langle \boldsymbol{b} \rangle$ are solenoidal we do have $\gamma_{iji} = \gamma_{ijj} = 0$.

Possible new instability mechanism

Even if the α-effect vanishes, it still seems that there can be a long-wavelength instability induced by the coupling terms.

Seek solutions proportional to e^{iKZ+ST}, so that $\langle \boldsymbol{u} \rangle_3 = \langle \boldsymbol{b} \rangle_3 = 0$.
Gives coupled two-dimensional algebraic equations (where $G^{(1)}_{ij} = \gamma_{i3j}$, $G^{(2)}_{ij} = \gamma_{3ij}$)

$$S\langle \boldsymbol{u} \rangle = 2i\tau K(\boldsymbol{G}^{(1)} + \boldsymbol{G}^{(2)})\langle \boldsymbol{b} \rangle, \quad S\langle \boldsymbol{b} \rangle = -2i\tau K(\boldsymbol{G}^{(1)} - \boldsymbol{G}^{(2)})\langle \boldsymbol{u} \rangle,$$

and so S^2 is an eigenvalue of the matrix

$$4\tau^2 K^2 \left((\boldsymbol{G}^{(1)} + \boldsymbol{G}^{(2)})(\boldsymbol{G}^{(1)} - \boldsymbol{G}^{(2)}) \right)_{ij}.$$

It would appear to be straightforward to find examples for which the eigenvalue S^2 is positive, implying instability of the basic state.

This would be a new generic mechanism for long wavelength instability, relying on coupling between the mean momentum and induction equations and on anisotropy in the basic flow statistics.

Implementation

General idea:

Calculate the coefficients of the α and Γ tensors by imposing *kinematic* and *uniform* fields and flows and solving the fluctuation equations.

Armed with the results of this small-scale calculation, we are in a position to determine the evolution of the mean fields.

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But does it always work?

Cases where it works well

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Suppose the basic state flow and field depend only on x, y and t (e.g. from a forced flow with a background field). This is an extension of the classic work of G.O. Roberts (1970).

Then one can consider perturbations of the form

$$\boldsymbol{u} = \hat{\boldsymbol{u}}(x, y, t) \exp(ikz), \qquad \boldsymbol{b} = \hat{\boldsymbol{b}}(x, y, t) \exp(ikz).$$

Here the long direction is in z, so $k \ll 1$. Averages are taken over the xy-plane, so we can perform two-dimensional calculations to tell us about the growth of long wavelength three-dimensional perturbations.

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We have considered two specific forcings, to drive simple (known) flows when there is no magnetic field (Courvoisier, H & Proctor 2010 *Proc. R. Soc.*):

$$\boldsymbol{U}_0 = \nabla \times (\psi \hat{\boldsymbol{z}}\ ) + w\hat{\boldsymbol{z}}\ \equiv \boldsymbol{U}_H + w\hat{\boldsymbol{z}}\ .$$

1. The AKA forcing (Frisch, She & Sulem 1987)

$$\boldsymbol{F} = \left(Re^{-1}\sqrt{2}\cos\left(y + Re^{-1}t\right), Re^{-1}\sqrt{2}\cos\left(x - Re^{-1}t\right), F_x + F_y \right).$$

2. The MW+ flow (Otani 1993)

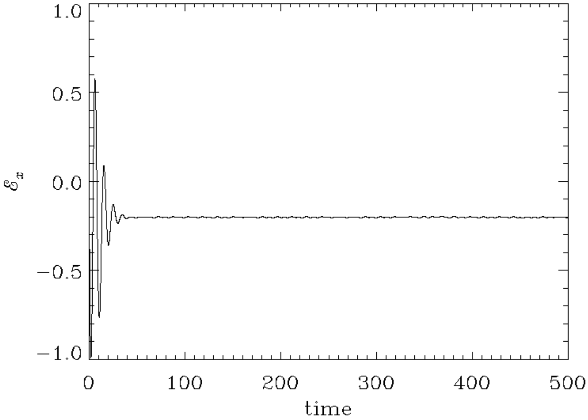
$$\psi = -w = \left(\cos x \cos^2 t - \cos y \sin^2 t\right).$$

Calculate the four tensors $\boldsymbol{\alpha}^B$, $\boldsymbol{\alpha}^U$, $\boldsymbol{\Gamma}^B$, $\boldsymbol{\Gamma}^U$ and hence the growth rate of any instability.

Calculating the averages

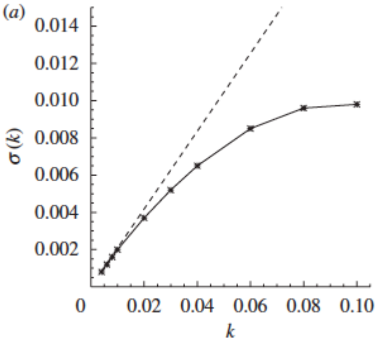
We impose uniform kinematic mean fields and flows in the xy-plane in order to calculate the components of the four tensors $\boldsymbol{\alpha}^B$, $\boldsymbol{\alpha}^U$, $\boldsymbol{\Gamma}^B$, $\boldsymbol{\Gamma}^U$.

For the cases we have looked at, the averages are well-behaved. For example, the cumulative time average of the x-component of $\langle \boldsymbol{U} \times \boldsymbol{b}' + \boldsymbol{B} \times \boldsymbol{u}' \rangle$:

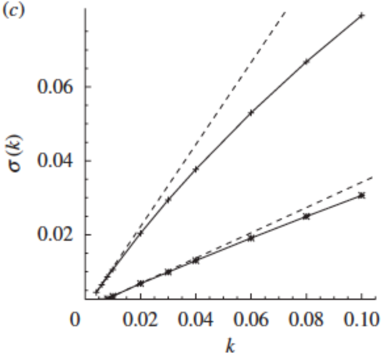


Comparison with the instability growth rate

For two-dimensional flows we can, independently, solve the three-dimensional problem and then compare the true growth rate with that derived from the mean field approach.



AKA flow



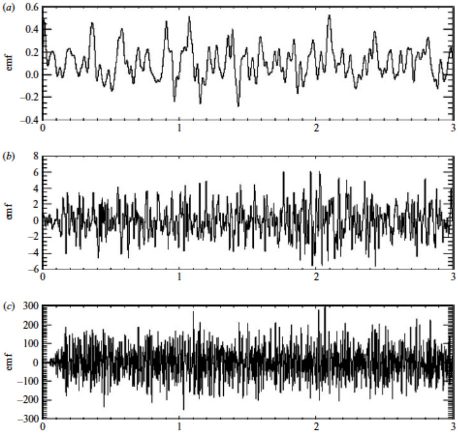
MW flow

(From CHP 2010.)

$$\sigma(k) = \Re\left(p(k)\right)$$

Possible Problems

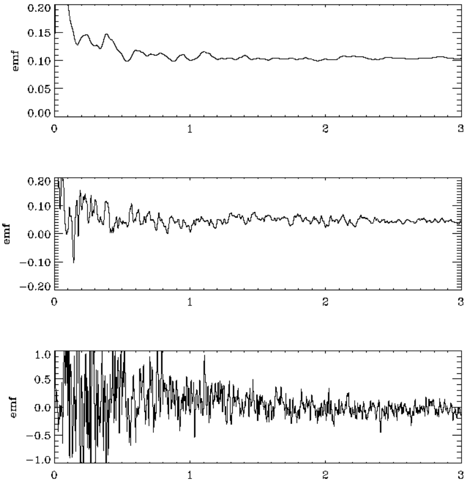
1. Mean value dominated by fluctuations, so averages are, at best, hard to compute:



Rotating convection: calculation of emf after imposition of weak uniform field. (*a*), (*b*) no small-scale dynamo; (*c*) small-scale dynamo. (From Hughes & Cattaneo 2008.)

Possible Problems

1 (contd). Even cumulative time averages sometimes inconclusive:



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We found this by considering certain types of ABC forcing.

Here we choose the forcing to drive the $1:1:1$ ABC flow defined by

$$\boldsymbol{U} = (\sin z + \cos y, \sin x + \cos z, \sin y + \cos x).$$

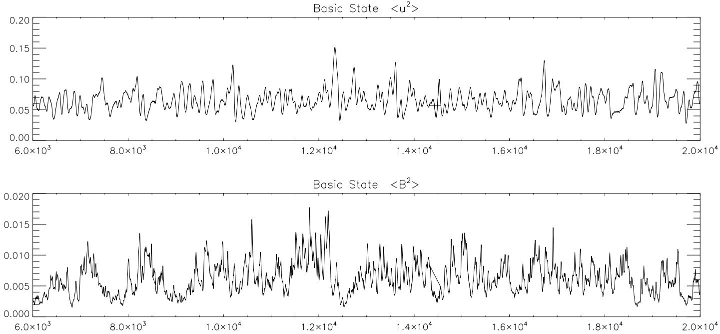
The flow is stable to the forcing for small enough Re.

For small values of Rm, the averages are well-defined, as for the $2D$ flows.

But there are problems at higher Re and Rm.

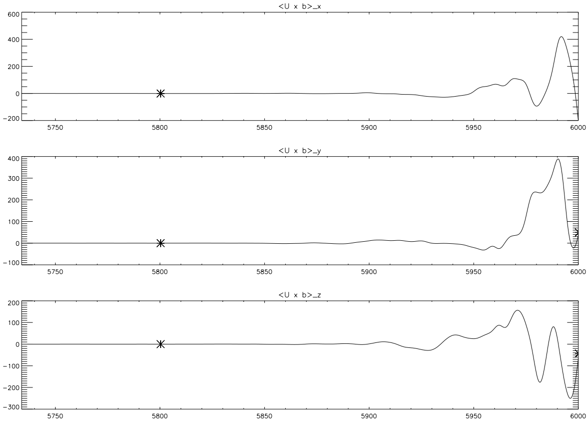
ABC forcing: $Re = Rm = 300$

Resulting flow is now no longer the ABC flow but is time-dependent and disordered.



Time series of the basic state flow $\vec{U}$ and field $\vec{B}$ for $Re = Rm = 300$.

ABC forcing: exponential growth of averages



Three components of $\langle \boldsymbol{U} \times \boldsymbol{b} \rangle$ versus time after imposition of $B_0 \hat{\boldsymbol{x}}$.

Response to the nonlinear problem

Rather than imposing kinematic mean flows and fields — which can lead to ill-defined mean quantities — we could instead impose *dynamic* flows and fields; i.e. all the nonlinear terms are retained.
We might expect that a small symmetry-breaking term would lead to a small change in mean quantities, even with large excursions.

So can we calculate the tensors $\boldsymbol{\alpha}^B$, $\boldsymbol{\alpha}^U$, $\boldsymbol{\Gamma}^B$, $\boldsymbol{\Gamma}^U$ from the linear limit of the nonlinear equations, which we presume has bounded solutions?

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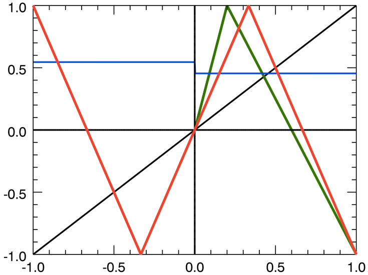
Let us consider this question by first examining some ostensibly (much) simpler problems.

We shall consider three one-dimensional maps $x_{n+1} = f(x_n)$:

- Tent map
- Cubic logistic map
- Lorenz map

Tent Map



$a = 3$ gives the symmetric map (red).
Asymmetric map (green) has $a = 5$.

$f(x)$ defined by:

$$
\begin{aligned}
f(x) &= -2-3x & -1 \le x \le -1/3 \\
&= 3x & -1/3 \le x \le 0 \\
&= ax & 0 \le x \le 1/a \\
&= \frac{1+a-2ax}{a-1} & 1/a \le x \le 1
\end{aligned}
$$

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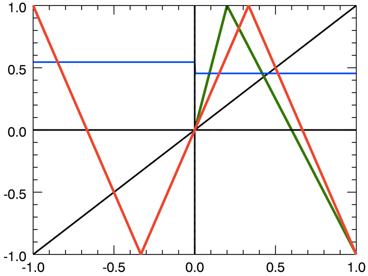

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Asymmetric map (green) has $a = 5$.
Probability density for iterations, $\rho(x)\mathrm{d}x$ (the *invariant measure*), given by Frobenius-Perron operator:

$$\rho(x) = \sum_i \frac{\rho(\tilde{x}_i)}{|f'(\tilde{x}_i)|}, \qquad \text{with} \quad \int \rho(x)\mathrm{d}x = 1.$$

Invariant measure is constant for the symmetric tent map. For the asymmetric map it is piecewise constant, $\rho = \rho_+$ for $x > 0$, $\rho = \rho_-$ for $x > 0$.
Blue line shows the invariant measure for $a = 5$.

Calculating $\rho(x)$

Each point x has three pre-images $\tilde{x}_i$ (two positive and one negative if $x > 0$). Then ρ_+ and ρ_- satisfy the two equations

$$\rho_+ = \frac{1}{3}\rho_- + \left(\frac{1}{a} + \frac{a-1}{2a}\right)\rho_+, \qquad \rho_- = \frac{2}{3}\rho_- + \frac{a-1}{2a}\rho_+.$$

These two relations are equivalent, so together with

$$\rho_+ + \rho_- = 1,$$

we obtain

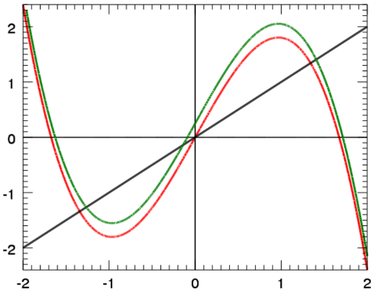
$$\rho_+ = \frac{2a}{5a-3}, \qquad \rho_- = \frac{3(a-1)}{5a-3}.$$

Mean value of x given by

$$\langle x \rangle = \rho_- \int_0^1 x \mathrm{d}x + \rho_+ \int_0^1 x \mathrm{d}x = \frac{1}{2}\left(\rho_+ - \rho_-\right) = \frac{3-a}{2(5a-3)}.$$

So here, small deviations in asymmetry (i.e. in $a-3$) lead to small (linear in $a-3$) perturbations to the mean.

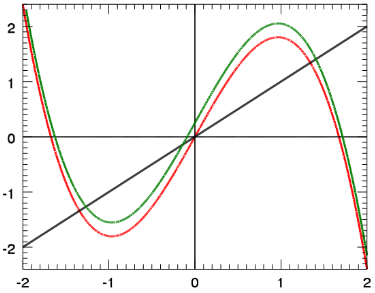
The Cubic Logistic Map



$f(x)$ defined by:

$$f(x) = \mu_0 + 2.8x - x^3$$

Map is into [-2,2] provided that μ_0 is sufficiently small.



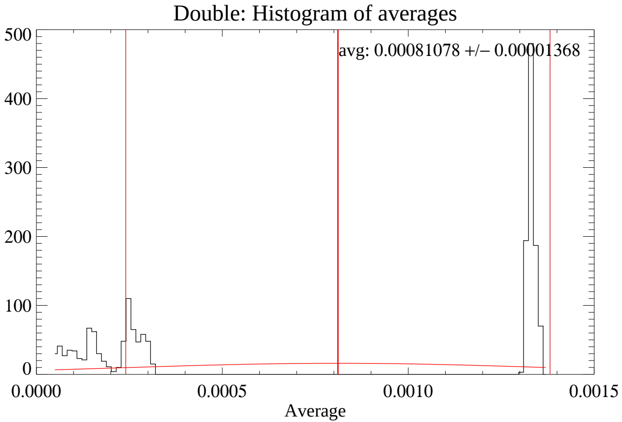
$f(x)$ defined by:

$$f(x) = \mu_0 + 2.8x - x^3$$

Map is into [-2,2] provided that μ_0 is sufficiently small.

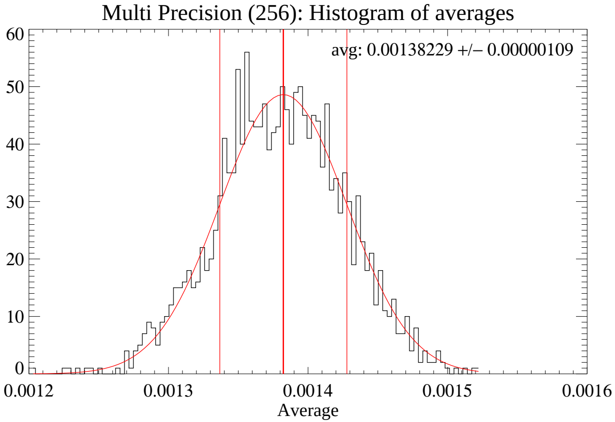
Not possible to calculate $\rho(x)$ analytically in this case.
What can we learn from numerical iterations of the map?

The Cubic Logistic Map: double precision iterations



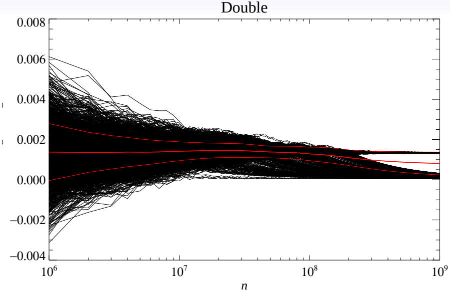
$\mu_0 = 10^{-6}$
2000 starting points
10^9 iterations
Double precision (52 bit mantissa)

The Cubic Logistic Map: multiple precision iterations

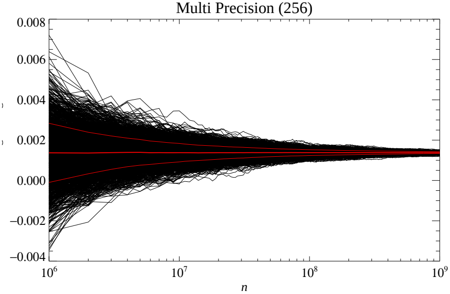


2000 starting points
10^9 iterations
Multiple precision (256 bit mantissa)
Red vertical lines show mean and $\pm$ standard deviation; red curve is associated Gaussian

The Cubic Logistic Map: Iterations

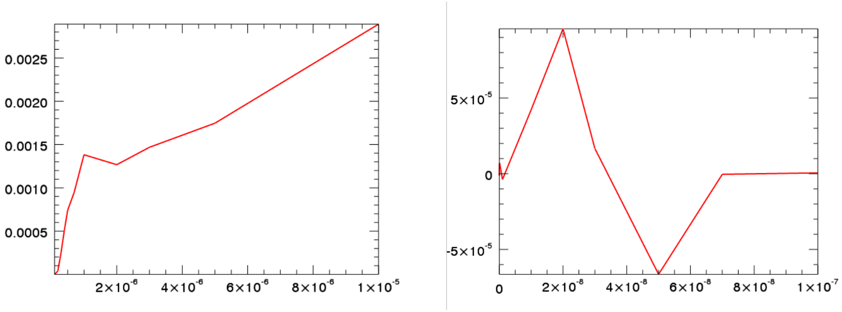


Inaccurate results with double precision



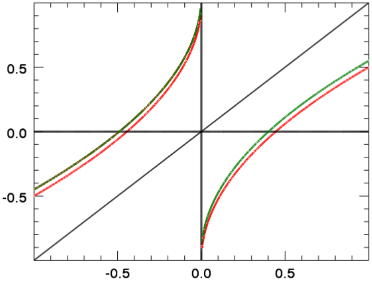
Multiple precision
2000 initial conditions, 10^9 iterations

The Cubic Logistic Map: Variation of $\langle x \rangle$ with μ_0



Complicated dependence of $\langle x \rangle$ on μ_0 — certainly no obvious linear dependence. Attributable to the existence of a dense set of periodic windows, leading to a highly complex invariant measure.

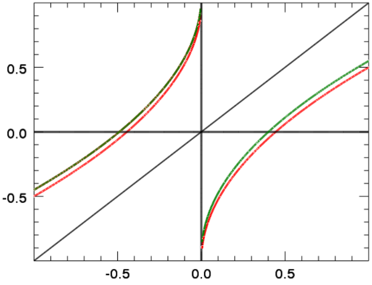
The Lorenz Map



$$f(x) = \mu_0 + \mathrm{sgn}(x)\left(-1 + 1.5\sqrt{|x|}\right)$$

No stable periodic orbits, so possibly more sensible results

The Lorenz Map



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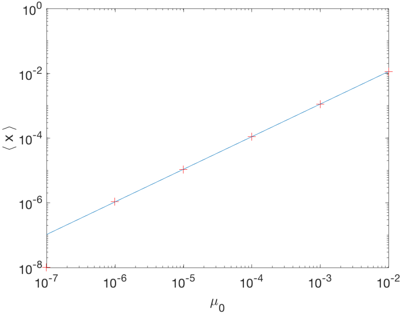

$\langle x \rangle$ versus μ_0
Excluding smallest value of μ_0, very good linear dependence (slope $= 1.007$).

Back to the MHD problem

If we now return to the MHD ABC problem, and impose *dynamical* fields and flows, is there a measurable response that is linear in the strength of the flow and field?

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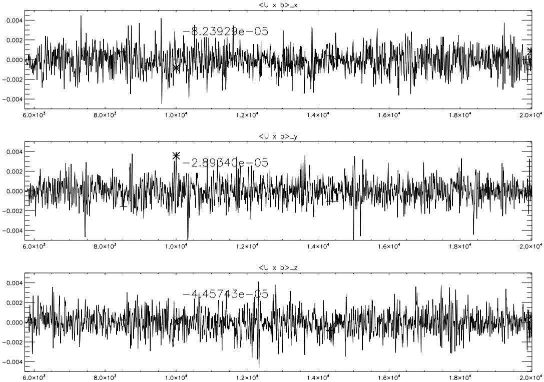
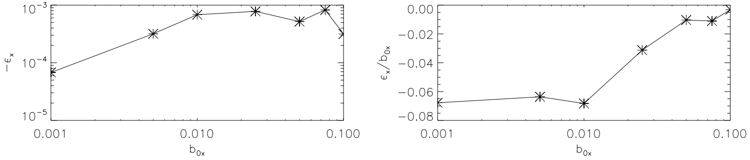


Three components of $\langle \boldsymbol{U} \times \boldsymbol{b} \rangle$ versus time.

Imposed field $\boldsymbol{B} = 10^{-3}\hat{\boldsymbol{x}}$; $Re = Rm = 300$.

Dynamic MHD: linear response?



e.m.f. versus imposed field strength

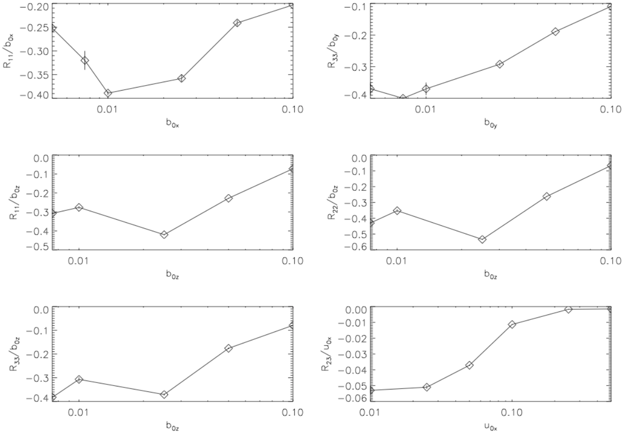
For sufficiently weak imposed magnetic fields, we can detect a linear response.

But one needs to get into the very weak field regime and integrate for very long times.

Dynamic MHD: another example

Forcing chosen to give non-zero $\boldsymbol{\alpha}$ and $\boldsymbol{\Gamma}$ tensors.:

$$\boldsymbol{F} = MW_{+} + 0.5 * ABC(1:1:1) \qquad (Re = 200, Rm = 500).$$



Certainly at these field strengths not yet a convincing linear response.

Conclusions

- The long wavelength instability of MHD states is determined by four tensors, $\boldsymbol{\alpha}^B$, $\boldsymbol{\alpha}^U$, $\boldsymbol{\Gamma}^B$, $\boldsymbol{\Gamma}^U$. Two of these are new, and they must all be considered in order to determine instability. The symmetry properties etc. of these tensors have not yet been fully explored.

- Calculating the tensors through the linear problem can lead to unbounded results. Can the nonlinear problem give a well-defined linear response? Examination of simpler iterative maps suggests that the answer is not straightforward (cf. Baladi 2014; Gottwald et al. 2016).

- Can speculate that with smooth invariant measures on the attractor there may be some hope of finding a linear response. However there is still the problem of the signal/noise ratio.

- Not totally clear how this carries through to complex systems. For our MHD problem, provided the flow is sufficiently complicated it seems that, for small enough imposed fields and flows, there is a measurable linear response. However the computational effort required to probe this regime is significant.

- Is this linear response what we need to apply the linear evolution theory? This may be testable in 2D.