Turbulence: the legacy of Leray, of Kolmogorov and of Uriel Frisch

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Uriel and turbulence

Memories of fifty years of turbulence studies.
Our student's time and after.
A (short) presentation of recent joint work with Martine Le Berre and Thierry Lehner: intermittency explained!

A quote to keep in mind

A quote to keep in mind when studying turbulence
"A great many people think they are thinking when they are rearranging their prejudices"
William James (US Pragmatist philosopher and psychologist, 1842 - 1910)
Quoted by P.H. Davidson in his book on Turbulence (Cambridge U. Press)

Weierstrass counterexample (1872)

Example of a periodic continuous function of t non differentiable almost everywhere (original notations)

$$f(t) = \Sigma_\nu b^\nu \cos(a^\nu \pi t),$$

Is continuous and non differentiable almost everywhere if a positive integer and if

$$ab > 1 + 3\pi/2.$$

The fundamental question in turbulence:
Do flows of incompressible fluids at large or infinite Reynolds number (namely at small or zero viscosity) present finite time singularities localized in space? Short presentation of the theoretical situation or are they continuous and non differentiable as Weierstrass counterexample?
Single-point records of velocity fluctuations display correlations between large velocities and large accelerations in full agreement with scaling laws derived from Leray-like equations (1934) for self-similar singular solutions to the fluid equations (Euler-Leray equations). Conversely, those experimental velocity - acceleration correlations are contradictory to the Kolmogorov scaling laws.
Moreover the so-called structure functions display a remarkable transition at increasing power of the fluctuation, in full agreement with what is found by supposing the flow as made of individual Leray-like singular events almost independent of each other.

Leray's singularities

The Euler-Leray equations for self-similar singular solutions of an inviscid incompressible fluid are derived from the Euler equations. The similarity exponents take into account either Kelvin's theorem of conservation of circulation or energy conservation (if energy is finite)
1) What are Euler-Leray equations ? + a strategy for an explicit (analytical) solution.
2) Amazing agreement between predictions of Euler-Leray with intermittency deduced from recordings of velocity fluctuations in Modane wind tunnel.
Dissipation by localized singularities in other settings: shock waves in compressible fluids, white caps of gravity waves, NLS focusing equation (joint work with Christophe Josserand and Sergio Rica)
Challenge (+ work in progress): put localized (space and time) dissipation in a coherent statistical framework.

Derivation of Leray's equations.1

In 1934 Jean Leray ("Essai sur le mouvement d'un fluide visqueux emplissant l'espace", Acta Math. **63** (1934) p. 193 - 248) published a paper on the equations for an incompressible fluid in 3D. He introduced many ideas, among them the notion of weak solution and also what problem should be solved to show the existence (or not) of a solution singular after a finite time with smooth initial data.
Leray assumed a solution of Navier-Stokes 3D blowing-up in finite time at a point, following self-similar evolution for reasonable initial data. Unknown yet if this solution exists, either for Euler and/or NS.

Derivation of Leray's equations.2

Euler equations (inviscid, incompressible, 3D):

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p,$$

and

$$\nabla \cdot \mathbf{u} = 0,$$

Leray looked (with viscosity added, Navier-Stokes equations) to self-similar solutions of the type:

$$\mathbf{u}(\mathbf{r}, t) = (t^* - t)^{-\alpha} \mathbf{U}(\mathbf{r}(t^* - t)^{-\beta}),$$

where t^* is the time of the singularity (set to zero), where α and β are positive exponents to be found and where $\mathbf{U}(.)$ is to be derived by solving Euler or NS equations.
That such a velocity field is a solution of Euler or NS equations implies $1 = \alpha + \beta$. The conservation of circulation in Euler equations implies $0 = \alpha - \beta$, and $\alpha = \beta = 1/2$. If one imposes instead that a finite energy in the collapsing domain is conserved, one must satisfy the constraint $-2\alpha + 3\beta = 0$, which yields $\alpha = 3/5$ and $\beta = 2/5$, the Sedov-Taylor exponents.

Derivation of Leray's equations.3

No set of singularity exponents can satisfy both constraints of energy conservation and of constant circulation on convected closed curves. $\alpha = \beta = 1/2$ if there are smooth curves invariant under Leray stretching.
Otherwise one has to take the Sedov-Taylor scaling, assuming that
1) the collapsing solution has finite energy,
2) no closed curve is carried inside the singular domain while keeping finite length and remaining smooth.

Derivation of Leray's equations.4

Introduce boldface letters such that $\mathbf{R} = \mathbf{r}(-t)^{-\beta}$. The Euler equations become the Euler-Leray equations for $\mathbf{U}(\mathbf{R})$:

$$-(\alpha\ \mathbf{U} + \beta\ \mathbf{R} \cdot \nabla \mathbf{U}) + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla P,$$

and

$$\nabla \cdot \mathbf{U} = 0$$

A general time dependence can be kept besides the one due to the rescaling of the velocity and distances by defining as new time variable $\tau = -\ln(t^* - t)$. This maps the dynamical equation into

$$\frac{\partial \mathbf{U}}{\partial \tau} - (\alpha \mathbf{U} + \beta \mathbf{R} \cdot \nabla \mathbf{U}) + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla P,$$

$$\nabla \cdot \mathbf{U} = 0$$

Equivalent to the original Euler equations.

Explicit solution of Euler-Leray equations: an outline

Euler-Leray equations in axisymmetric geometry with swirl and dependence on τ (work in progress + Pomeau-Le Berre in Arxiv):
1) Start from a localized solution of steady localized Euler equation by solving Hicks (or Bragg-Hawthorne) equations. Because this has finite energy one takes Sedov-Taylor exponents.
2) Because steady Euler equations are invariant under arbitrary dilations of amplitude or argument (being homogeneous of order 2 and invariant under dilation of coordinates) one can assume that the solution of Hicks equation has very large amplitude.
3) This makes the (linear) streaming term added by Leray arbitrarily small compared to the leading order term which is quadratic.
4) Solving Euler-Leray by perturbation one meets two solvability conditions because of the two dilation symmetries of the steady Euler equations. They can be satisfied by adding two small oscillations with arbitrary amplitudes or by tuning free coefficients in the background solution of Hicks equation.

Is it possible to "observe" Euler-Leray singularities?.1

Our motivation for working on Euler-Leray singularities is their possible connection with the phenomenon of intermittency in high Reynolds number flows. This raises several questions:
1. What is specific to Leray singularities compared to other schema for intermittency?
2. What would be specific of an Euler-Leray singularity in time records of single point velocity in a large Reynolds number flow ?
3. What are precisely the consequences of the occurrence of Leray-like singularities on the statistics of a turbulent flow?

Is it possible to "observe" Euler-Leray singularities?.2

Point 1 : If intermittency is caused by Leray-like singularities, they should yield strong positive correlation between singularities of the velocity and of the acceleration. This is what is observed.
Compared to scaling prediction derived from Kolmogorov-like exponents this (positive) correlation is a strong indication of the occurrence of singularities near large fluctuations. Moreover Kolmogorov theory extended to dissipative scales excludes exponents of the singularity of the velocity fluctuations vs distance which is less than $1/3$: otherwise dissipation is divergent everywhere in space, clearly impossible.
The only way-out is to have dissipative events at random points in space and time in the limit of large Reynolds number, instead of being spread uniformly in space and time (as singularities of the derivative in the counter example of Weierstrass, 1872).

Euler-Leray singularities and intermittency.1

Kolmogorov K41 theory is based upon the idea that turbulent fluctuations at very large Reynolds number (where the effect of viscosity is formally small) depend on the power dissipated in the turbulent flow per unit mass, ϵ.
Kolmogorov theory is successful for predicting the spectrum of velocity fluctuations (Kolmogorov-Obukhov spectrum $k^{-5/3}$) but is contradicted by intermittency. Because of it the fluctuations fail to satisfy the relationship predicted by Kolmogorov between the velocity fluctuation and the distance between two points of measurement. Using the scaling law with ϵ, one finds $< |u(r_0 + r, t) - u(r_0, t)|^3 >\sim (\epsilon r)$ when the distance r is in the (wide) range between the largest scales and the length scale short enough to make the viscosity relevant. If applied to arbitrary power n this predicts that, as r gets smaller and smaller, the amplitude of the velocity fluctuation decreases, not what is observed. K41 scaling fails badly as soon as $n > 4$.
Statistical theory based on random occurrence of Leray-like singularities (see later).

Euler-Leray singularities and intermittency.2

We have very long and high quality records of velocity fluctuations in the high-speed wind tunnel of Modane in the French Alps, obtained by hot-wire anemometry (Yves Gagne et al. 1998), and all sorts of correlations can be studied.
Suppose the large bursts of velocity are due to Euler-Leray singularities. It means that $u(r,t)$ scales like $(-t)^{-\alpha}$ as t tends to zero (0 taken arbitrarily as the instant of the singularity). The acceleration γ (time derivative of Eulerian u) is of order of $(-t)^{-(1+\alpha)}$ as t tends to zero. Therefore near the singularity both the velocity and the acceleration diverge, this last one the most strongly and in this large burst u^3 is of order γ if conservation of circulation is taken:

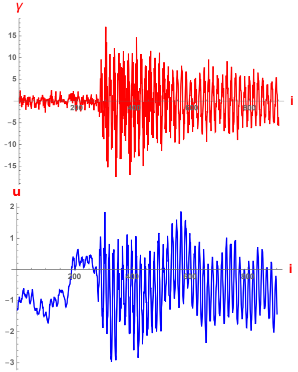
$$u^3 \sim \Gamma\gamma$$

The multiplicative constant is of the order of a "typical" value of the circulation. With the Sedov-Taylor exponents, on has instead:

$$u^8 \sim E\gamma^3$$

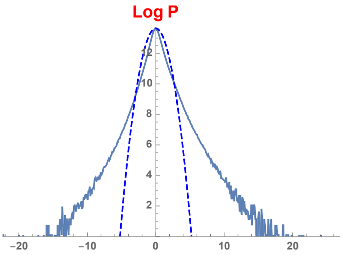
where E is the energy inside the collapsing domain.

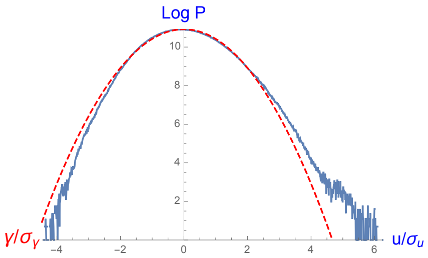
burst from Modane 2014; $\gamma(t)$ (red); $u(t)$ (blue)



$\gamma/g = 56000$; (Maximum ratio $\gamma/g = 10^6$ for Modane-2014 ; and $\gamma/g = 6000$ for Modane-1998) g acceleration of gravity.

Gaussian Statistics?

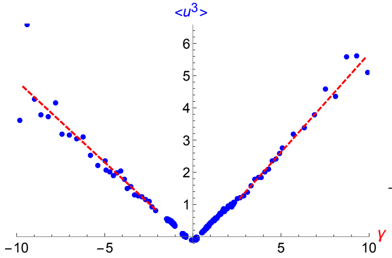


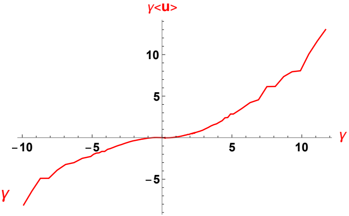


Non Gaussian acceleration

Slightly non Gaussian velocity

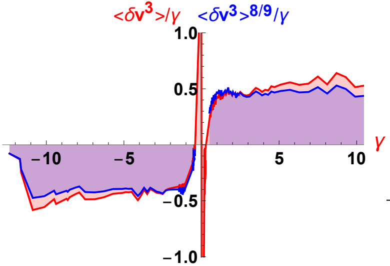
Scaling relations : $u^3 = \Gamma\gamma$ or $u\gamma \sim \epsilon$?

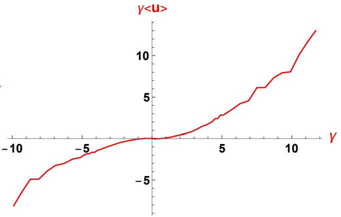




Scalings Leray/circulation: $u^3 = \Gamma\gamma$;
Scaling Kolmogorov $u\gamma \sim \epsilon$: invalid

Circulation scaling vs Sedov-Taylor scaling vs Kolmogorov scaling





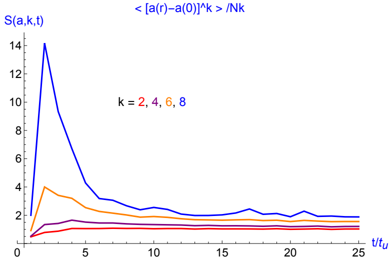
Scalings / circulation (left red): $u^3 \sim \Gamma\gamma$
Scalings / energy: Sedov-Taylor (left blue) $u^8 \sim E\gamma^3$;
Scaling Kolmogorov (right) $u\gamma \sim \epsilon$ on the right
Notice: Taylor frozen turbulence does not apply because the large velocity fluctuations are of the same order as mean velocity.

Sketch of a statistical theory based on the random occurrence of Leray singularities.1

$$\mathcal{M}_n(r) = \int \mathrm{d}q\, \nu_s(q) \int \mathrm{d}r_0 \int \mathrm{d}t\, |u_s(r + r_0, t|q) - u_s(r_0, t|q)|^n$$

where $\mathcal{M}_n(r) =< |u(r + r_0) - u(r_0)|^n >$ with $u_s(r, t|q)$ Leray-like solution singular at $t = r = 0$. Parameter q is for symmetries, and eventually a multiplicity of different solutions, $\nu_s(q)$ is the density of singularities in space-time. Two sources of dependence with respect to r: the phase space part (i.e. the volume $\mathrm{d}^3 r_0 \mathrm{d}t$ at small r) and the singular dependence of u_s. If n is less than a critical value depending on the exponents of the Leray-like solution, $\mathcal{M}_n(r)$ tends smoothy to zero whereas it diverges at $r \to 0$ if n is larger than a critical value. This is in very good agreement with Modane's data by taking the acceleration instead of u. This sharp dependence of $\mathcal{M}_n(r)$ near $r = 0$ is a direct consequence of the existence of singular solutions in real turbulent flows.
Everything in this approach is related to specific solutions of the Euler equations + input of $K - \epsilon$ theory to get $\nu_s(q)$

Sketch of a statistical theory based on the random occurrence of Leray singularities.2



blue: $n = 8$
purple: $n = 4$
red: $n = 2$
Notice the very sharp difference between the behavior of $\mathcal{M}_n(r)$ for small r as n gets bigger. For "larger" value of r the two singularities at $r + r_0$ and r_0 become independent and $\mathcal{M}_n(r)$ becomes constant.

Sketch of a statistical theory based on the random occurrence of Leray singularities.3

The explanation of this difference of behaviors as n increases relies on the estimate of the contribution of singular events to $\mathcal{M}_n(r)$, assuming first that those events follow a Leray-like law of self-similarity and then that the solution of the Euler-Leray equation is linearly stable, or equivalently that Leray-like singularities have a nonzero basin of attraction in phase space of initial conditions (perhaps a too strong condition-see remarks below and coming Arxiv paper). If one makes the first assumption, one finds that near $r = 0$ for the acceleration:

$$\mathcal{M}_n(r) \sim r^{3+1/\beta - n(\alpha+1)/\beta}$$

The first contribution to the exponent comes from the volume of physical phase space $\mathrm{d}^3 r_0 \mathrm{d}t$, the other, proportional to n from the divergence of the self-similar solution at $r = t = 0$. As n increases the exponent, as observed, changes from positive (decay of $\mathcal{M}_n(r)$ as r tends to zero) to negative (growth as r tends to zero, except for a round-off by viscosity very near $r = 0$).

Sketch of a statistical theory based on the random occurrence of Leray singularities.4

However, compared to the experimental values of the exponents the estimated exponents, when positive, are too big. This is because the parameter q related to the dilation invariance of the Euler equation depends on time τ and ultimately on viscosity, which amounts to add a contribution to u_s decaying like a power of τ. This takes into account that at very short distances viscosity becomes relevant and explains why the Euler-exponent overestimates the growth of $\mathcal{M}_n(r)$ at small r.
Another possibility is that α belongs to a continuous spectrum.