Phase dynamics in convective structures :
from small scale instabilities
to
large scale time-dependence

Festival de Théorie 2019

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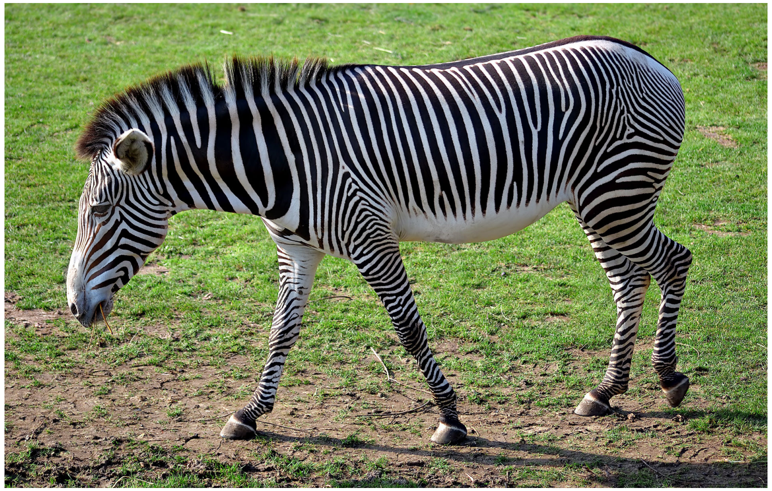
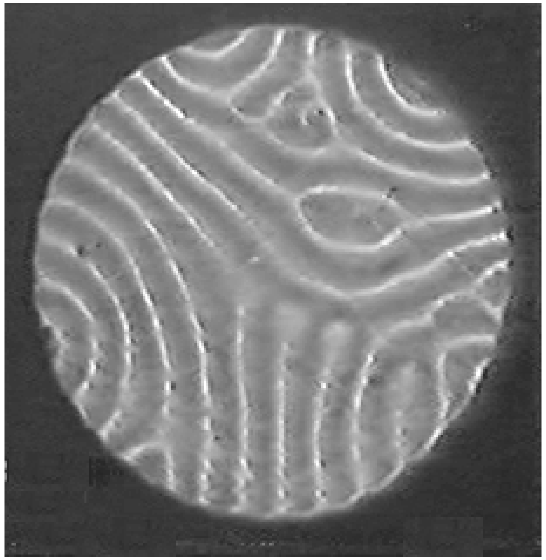
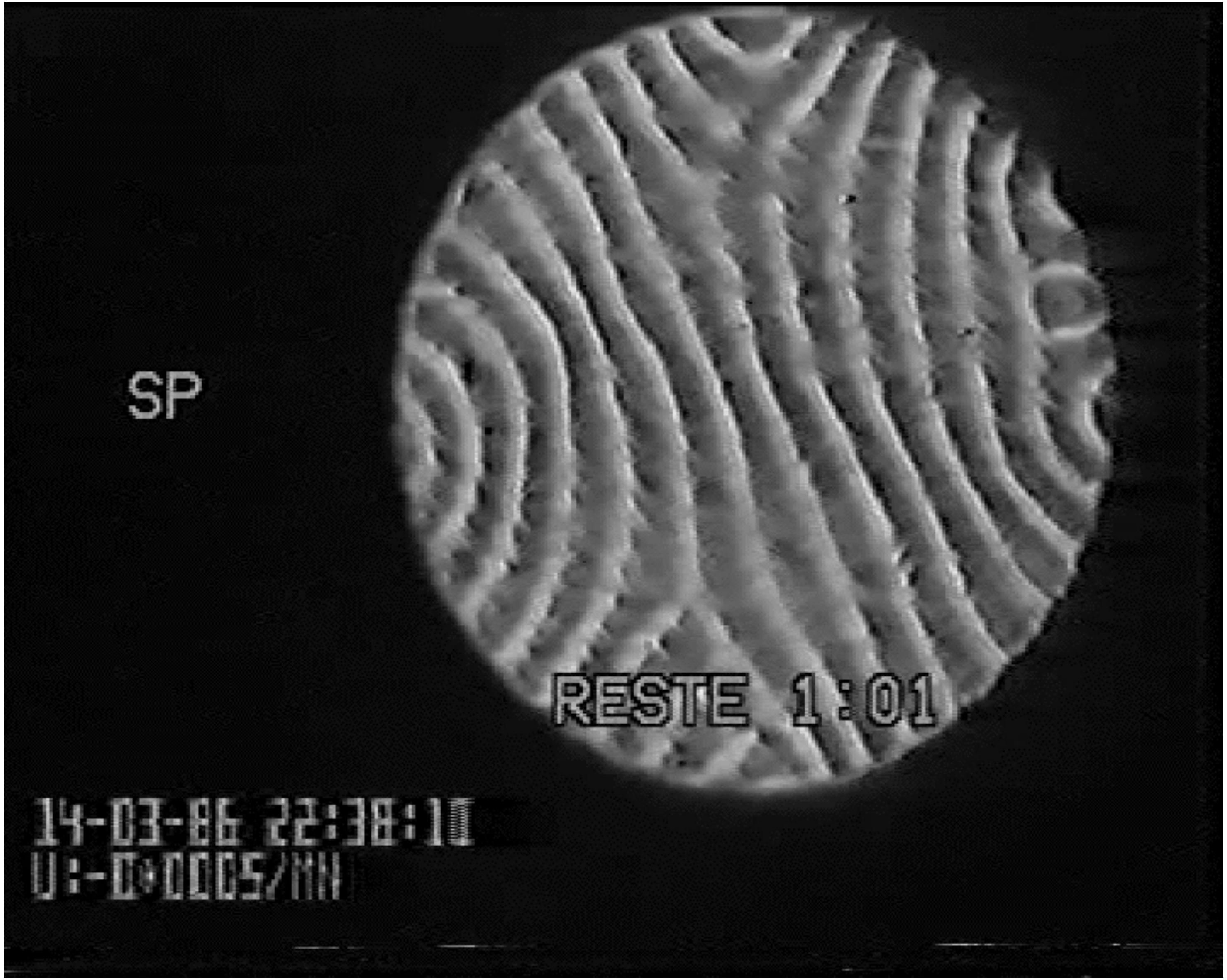
From patterns to phase





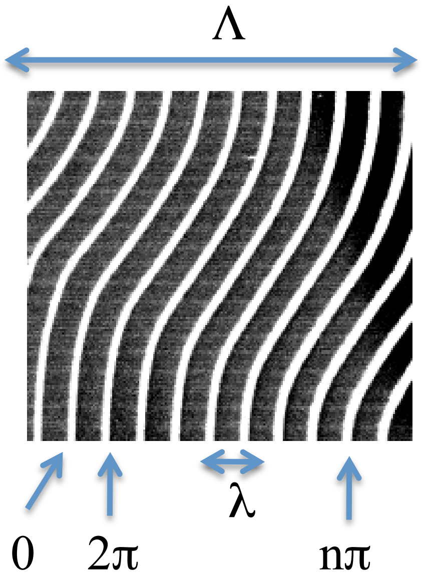
From patterns to phase

Phase turbulence



From patterns to phase

Universal features



in between bright lines : repetitive local structure

↓ ↓

lines : iso-phase φ — modulated periodic functions of φ

↓

$w = A \exp(i\varphi) + cc +$ harmonics

$w(\mathbf{r},t)$: real physical variable
$A(\mathbf{r},t)$: complex amplitude
$\varphi(\mathbf{r},t)$: real phase variable
cc : complex conjugate

Scales

- short : distance between iso-φ lines : local wavelength $\lambda = 2\pi/k$
- large : modulation length scale Λ

$\Lambda >> \lambda$
except at defects

Origin of patterns : by instability :

uniform basic state $\xrightarrow{\text{instability}}$ pattern formation

Underlying symmetries

- translational : $w(\mathbf{r} + \mathbf{r_0}, t)$: solution $\forall\, \mathbf{r_0}$
- galilean : $\left.\begin{array}{l} w(\mathbf{r} + \mathbf{V_0}\, t, t) \\ \mathbf{V} + \mathbf{V_0} \end{array}\right\}$ solution $\forall\, \mathbf{V_0}$

Amplitude : |A|

instability ↓

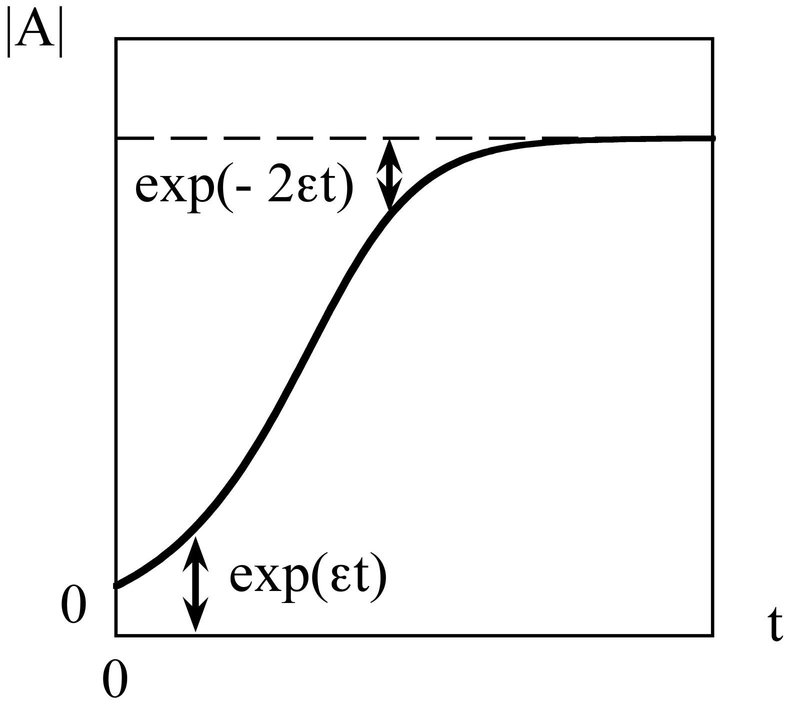
basic state : $|A|=0$

restabilized state : $|A|=|A_S| > 0$

Ginzburg-Landau equation

$$\frac{d|A|}{dt} = \epsilon|A| - |A|^3$$

↑ instability ↑ saturation



Time scale :

$$\tau_A = 1 / \varepsilon$$

Phase mode

Slow/rapid variables

Phase : φ

$$\left.\begin{array}{l}\text{uniform translation by } \mathbf{r}_0\\ \text{of a steady solution } w_0\end{array}\right\} \quad \left.\begin{array}{l}\varphi \longrightarrow \varphi+\varphi_0\\ w_0 \longrightarrow w_0(\mathbf{r}+\mathbf{r}_0) : \text{steady solution}\end{array}\right\} \quad \begin{array}{c}\text{uniform phase shift}\\ \varphi_0\\ \text{steady}\end{array}$$

$\updownarrow$

translational invariance

$$\left.\begin{array}{r}\nabla\varphi_0=0\\ \Lambda=\infty\end{array}\right\} \frac{\partial\varphi_0}{\partial t}=0$$

φ : **neutral** mode for translational modulation : $\Lambda = \infty$; $\tau_\varphi = \infty$

φ : **slow** mode for **long** wavelength modulation : $\Lambda >> \lambda$; $\tau_\varphi \approx \Lambda^2$

Slaved mode : |A|

$$\left.\text{long wavelength modulation : } \Lambda^2 >> 1/\varepsilon : \tau_\varphi >> \tau_A\right\} \quad \begin{array}{l}|A| : \text{rapid mode}\\ \varphi : \text{slow mode}\end{array}$$

→ adiabatic elimination of A : $A \equiv A_s[(\nabla^i\varphi)_{i\geq 0}]$ with $\frac{\partial A_s}{\partial t} = 0$

→ $$\frac{\partial\varphi}{\partial t} = f[(\nabla^i\varphi),(\nabla^j A)] \equiv F[(\nabla^i\varphi)]$$

Phase mode Phase diffusion

1d : $\mathbf{r} = x\,\mathbf{e}_x$

long wavelength limit :

expansion in $\frac{\partial\varphi}{\partial x}$

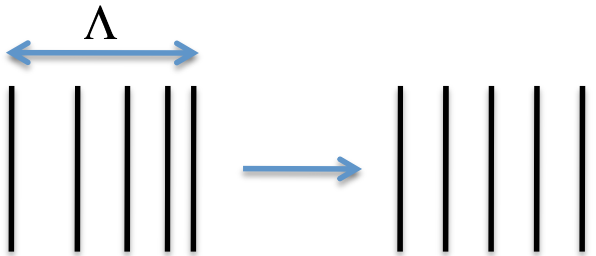
$$\left.\begin{array}{l}\\ \\ \end{array}\right\} \quad \frac{\partial\varphi}{\partial t} = a_0\varphi + a_1\frac{\partial\varphi}{\partial x} + b_{1,1}(\frac{\partial\varphi}{\partial x})^2 + a_2\frac{\partial^2\varphi}{\partial x^2} + o(\frac{\partial\varphi}{\partial x})$$

- translational invariance : phase origin arbitrary : $a_0 = 0$
- parity symetry : x → -x : $a_1 = 0$
- uniform patterns are steady : $\frac{\partial^2\varphi}{\partial x^2} = 0 \rightarrow \frac{\partial\varphi}{\partial t} = 0$ → $b_{1,1} = 0$

At the dominant order : $\frac{\partial\varphi}{\partial t} = D_{//}\frac{\partial^2\varphi}{\partial x^2} + o(\frac{\partial\varphi}{\partial x})$ → phase diffusion

Actually : $D_{//} \equiv D_{//}(k)$ with $k = |\frac{\partial\varphi}{\partial x}|$ → non-linear phase diffusion

$D_{//} > 0$:



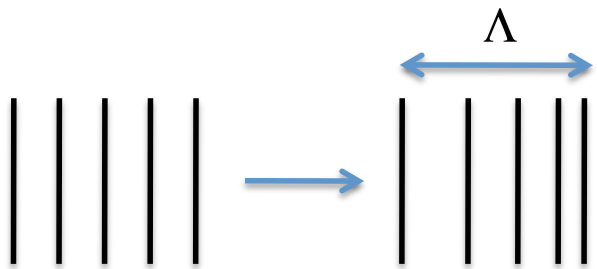
$\tau \approx \Lambda^2 / D_{//}$

Y. Pomeau, P. Manneville, J.Physique-Lettres 40,609,1979

Phase mode

Phase instabilities

$D_{//} < 0$:



$\tau \approx \Lambda^2 / D_{//}$

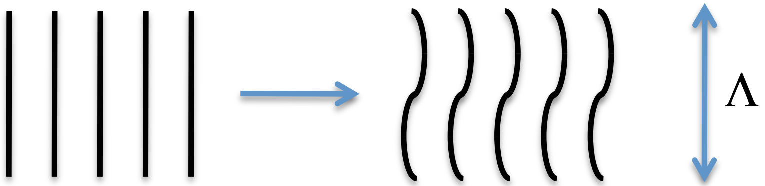
Eckhaus instability

Same on the y-direction :

anisotropic non-linear phase diffusion :

$$\frac{\partial \varphi}{\partial t} = D_{//} \frac{\partial^2 \varphi}{\partial x^2} + D_{\perp} \frac{\partial^2 \varphi}{\partial y^2} + o(|\nabla \varphi|)$$

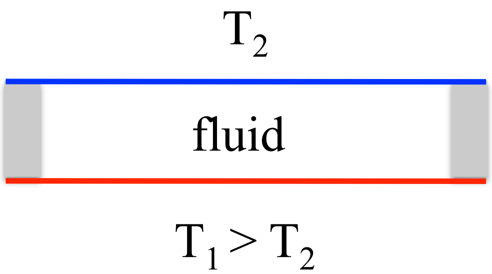
$D_{\perp} < 0$

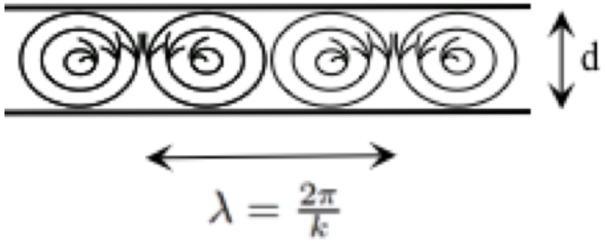


Zig-zag instability

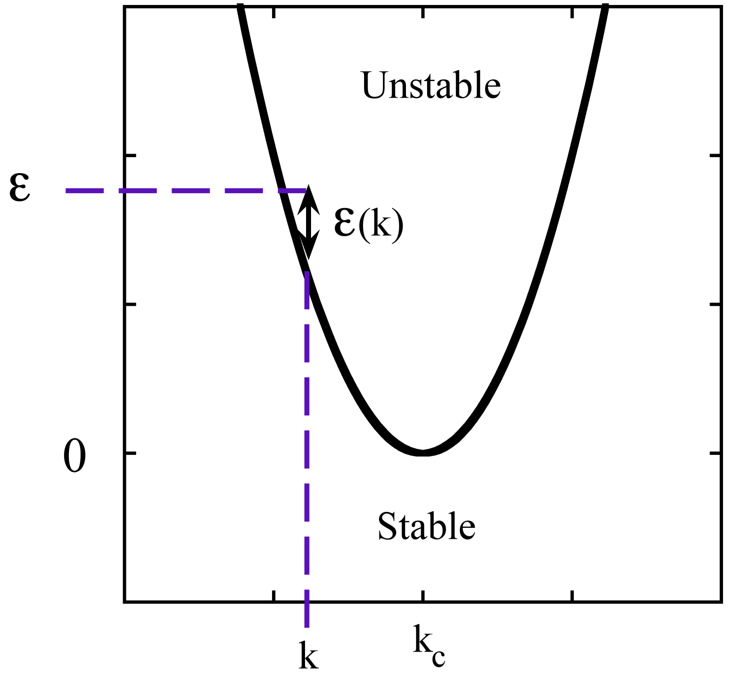
Phase dynamics in convective structures

Convective rolls





$$\varphi = kx$$



$$w_k = A_k \exp(ikx) + cc + harmonics$$

$$\frac{d|A_k|}{dt} = \epsilon(k)|A_k| - |A_k|^3$$

$$\epsilon = \frac{Ra - Rac}{Rac}$$

onset : ε=0

$$\epsilon(k) = \epsilon - \xi_0^2(k - k_c)^2 + h.o.t.$$

$\tau_{Ak} \approx 1/\varepsilon(k)$

$\tau_{\varphi} \approx \Lambda^2 / D$

phase diffusion valid for $\tau_{\varphi} >> \tau_{Ak}$

↳ $\Lambda^2 >> D / \varepsilon(k)$

Phase dynamics in convective structures

Amplitude equation

Oberbeck-Boussinesq equations {
- Navier-Stokes
- Fourier
- Incompressibility
- State equation ρ(P,T)

} $$\frac{\partial \mathbf{u}}{\partial t} = \mathcal{N}(\mathbf{u})$$

Stability of a steady state $\mathbf{u}_s$: $$\mathbf{u} = \mathbf{u}_S + \tilde{\mathbf{u}}$$

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} = \mathcal{L}(\tilde{\mathbf{u}}) + o(\tilde{\mathbf{u}})$$

$$\mathcal{L}(.) = \frac{\partial \mathcal{N}}{\partial \mathbf{u}}\Big|_{\mathbf{u}_S}$$ Jacobian

↓

linear spectrum : $$\sigma_k = \epsilon(k)$$

Phase dynamics in convective structures

Amplitude equation

Amplitude equation : $\tilde{\mathbf{u}}_k = A\exp(ik_c x)\,\mathbf{e} + cc + harmonics$

ansatz : $A \approx \epsilon^{1/2}$

Perturbative expansion in $\eta = \epsilon^{1/2}$: $\tilde{\mathbf{u}}_k = \sum_1^\infty \tilde{\mathbf{u}}_k^i \longrightarrow \mathcal{L}(\tilde{\mathbf{u}}_k^i) = N_i[\tilde{\mathbf{u}}_k^i, j < i]$

translational invariance implies : $\mathcal{L}(\frac{\partial \mathbf{u}_S}{\partial x}) = \mathbf{0} \longrightarrow Ker[\mathcal{L}] \neq \{\mathbf{0}\}$

Avoid resonance : $< \frac{\partial \mathbf{u}_S}{\partial x} \mid N_i[\tilde{\mathbf{u}}_k^i, j < i] >= \mathbf{0}$ → amplitude equation

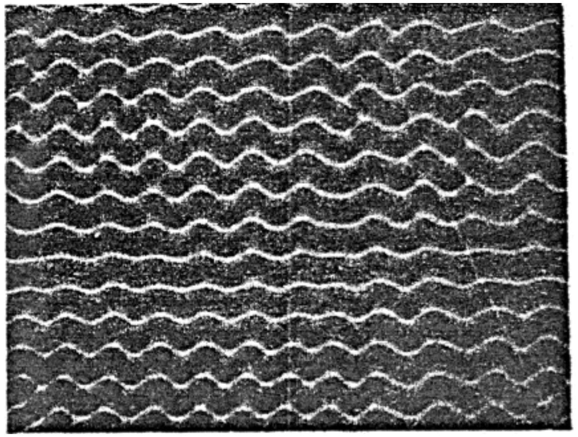
Newell-Whitehead-Segel : $$\frac{\partial A}{\partial t} = \epsilon A + \frac{\partial^2 A}{\partial X^2} - A|A|^2 + h.o.t.$$

→ contains the phase equation $Ae^{ik_c x} = |A|e^{i\phi}$
but perturbative method
enables an explicit determination of $D_{//}$ and $D_\perp$

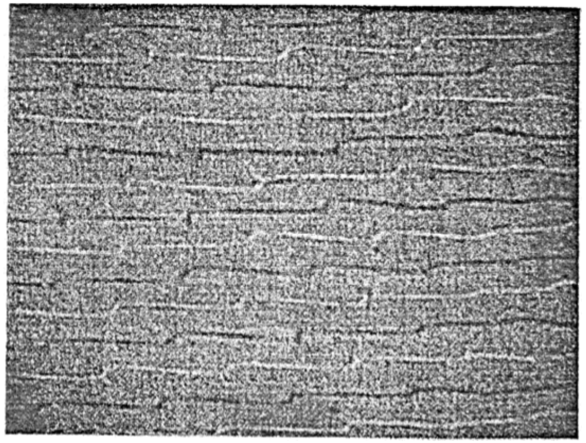
A.C.Newell, J.A.Whitehead, J.FluidMech.38,279,1969
F.H. Busse, Rep.Prog.Phys.38,203,1969

Instabilities

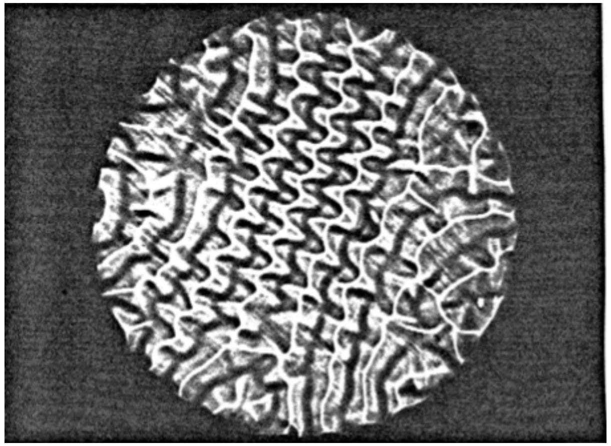
Instabilities of uniform roll patterns



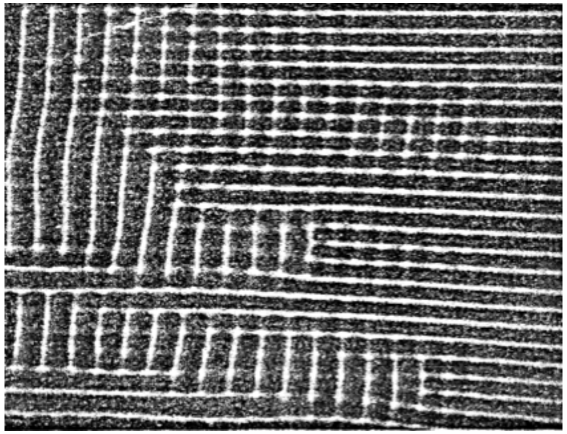
Zig-zag instability



Skewed-varicose instability



Oscillatory instability

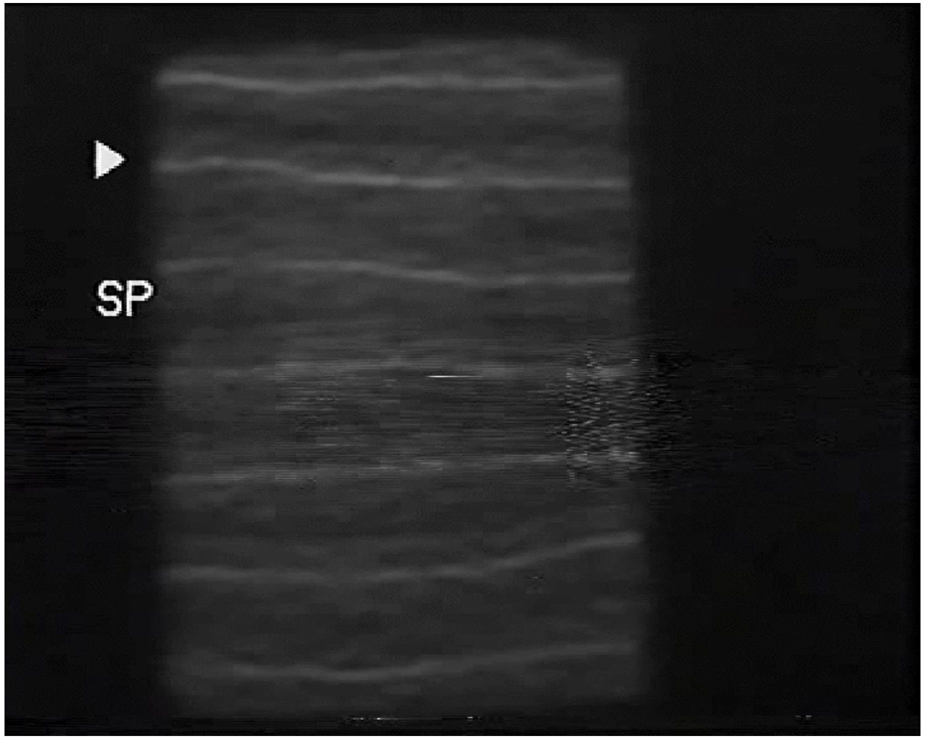


Cross-rolls instability

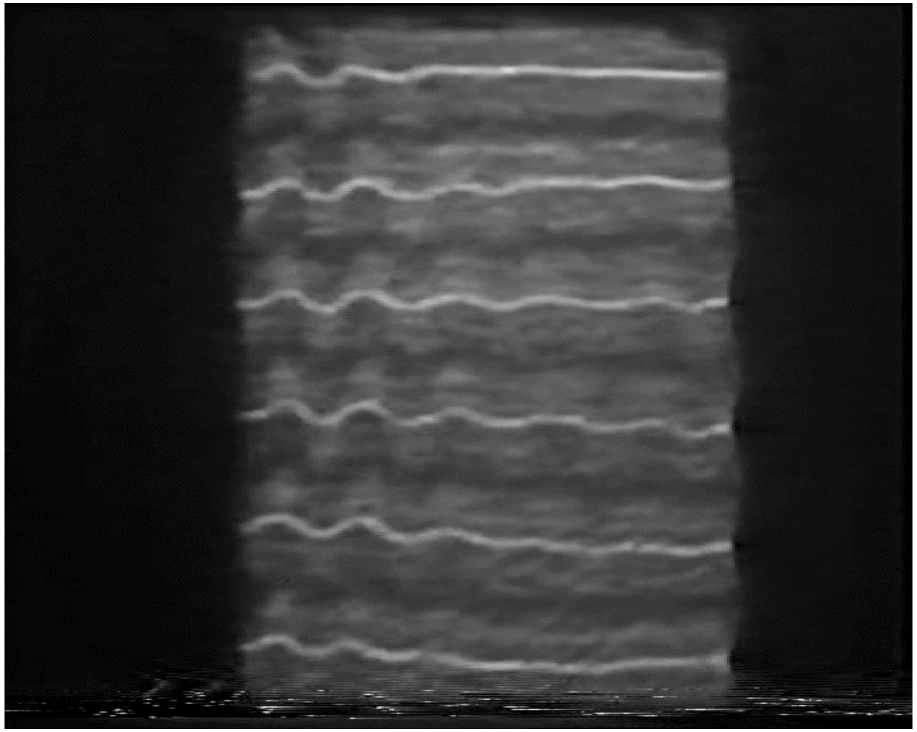
F.H. Busse, R.M. Clever, J.FluidMech.102,61,1981
F.H. Busse, Rep.Prog.Phys.41,1929,1978

Phase dynamics in convective structures

Instabilities



Skewed-varicose instability



Oscillatory instability

F.H. Busse, R.M. Clever, J.FluidMech.102,61,1981
F.H. Busse, Rep.Prog.Phys.41,1929,1978

Phase dynamics in convective structures

Instabilities

Uniform roll patterns :
stable on a closed domain

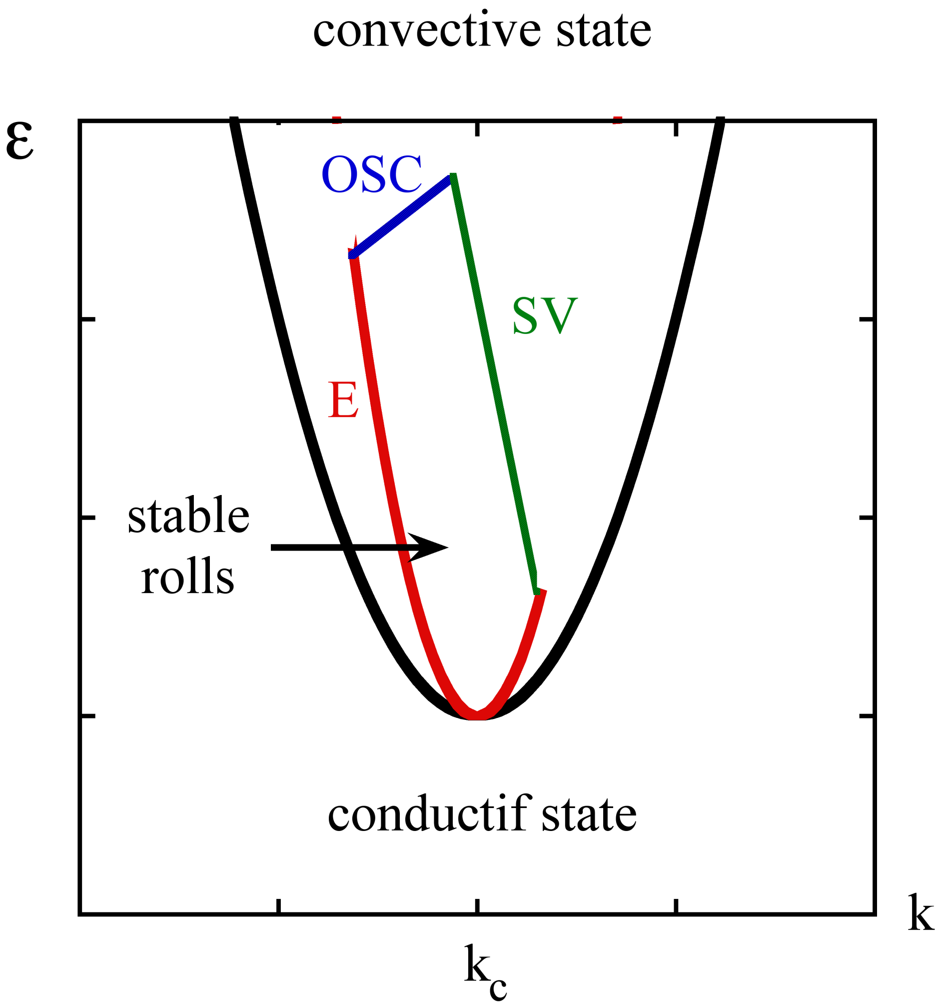
→ Busse balloon

NWS equation : variational

potential : $F \equiv F(A, \bar{A})$

$$\frac{\partial A}{\partial t} = \frac{\delta F}{\delta \bar{A}} \qquad \frac{dF}{dt} \leq 0$$

Fate of unstable patterns :
restabilize on steady patterns
made of nearly straight patterns
with defects



F.H. Busse, R.M. Clever, J.FluidMech.102,61,1981
F.H. Busse, Rep.Prog.Phys.41,1929,1978

Phase dynamics in convective structures

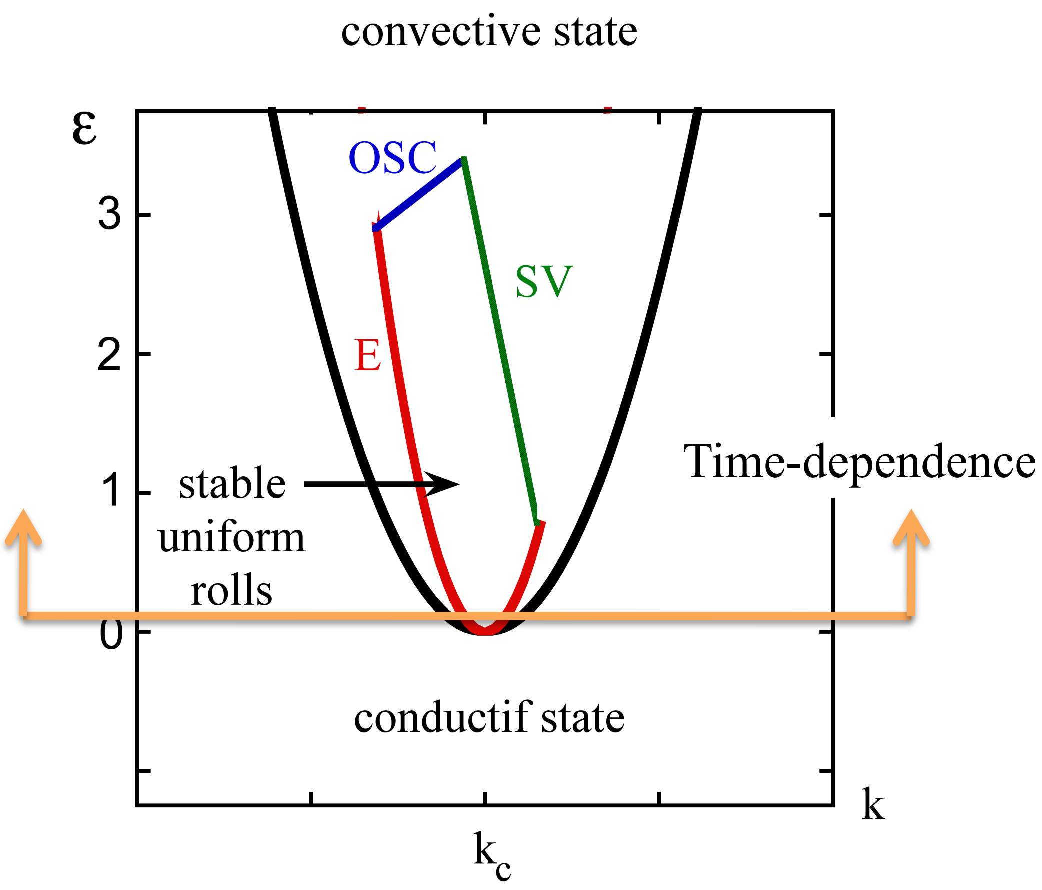
Permanent dynamics

Ahlers, Behringer, 1978

Liquid He : heat flux

- time dependence as soon as $\varepsilon = 0.09$
- erratic signal for $\varepsilon > 0.8$

No visualization



Why no restabilization on stable states ?

G. Ahlers, R.P.Behringer, PRL40,712,1978

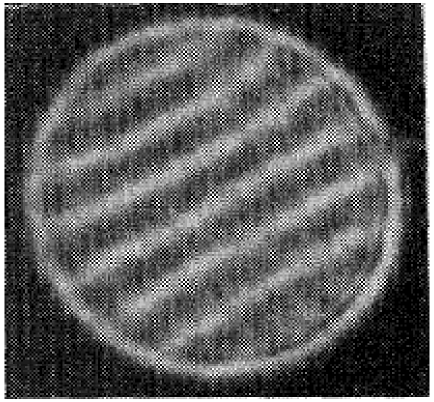
Phase dynamics in convective structures

Permanent dynamics

Convection in gaz : same Prandtl number than liquid He : Pr=0.7

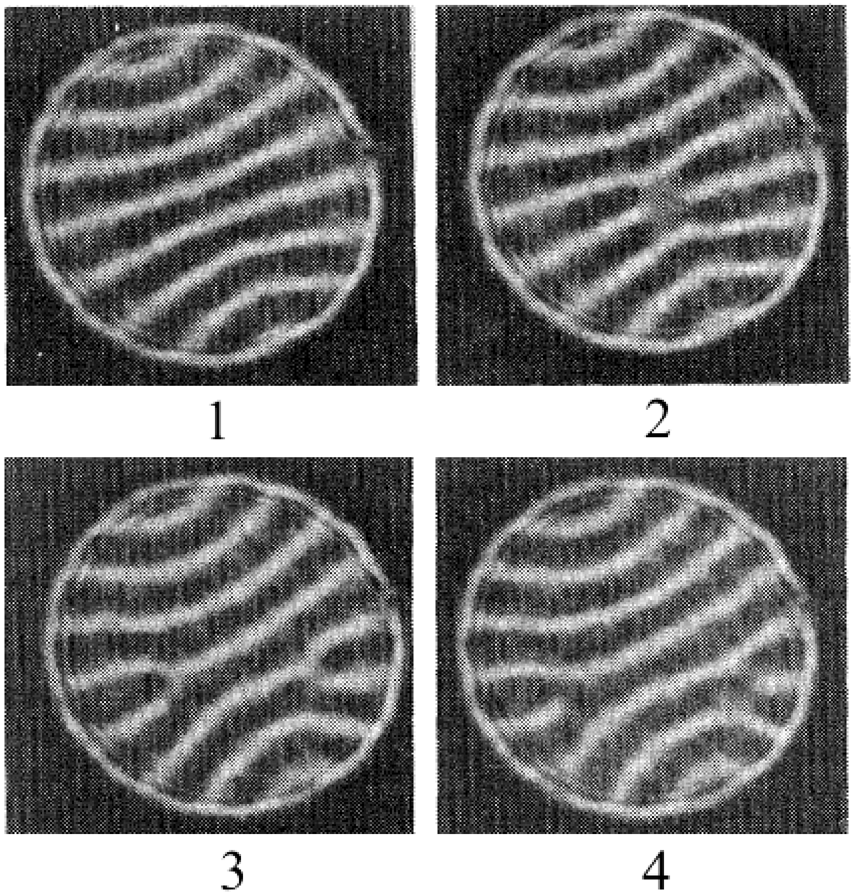
Increase of pressure → visualization

$\varepsilon = 0.05$



steady

$\varepsilon = 0.14$



A. Pocheau, V. Croquette, P. Le Gal, PRL55,1094,1985

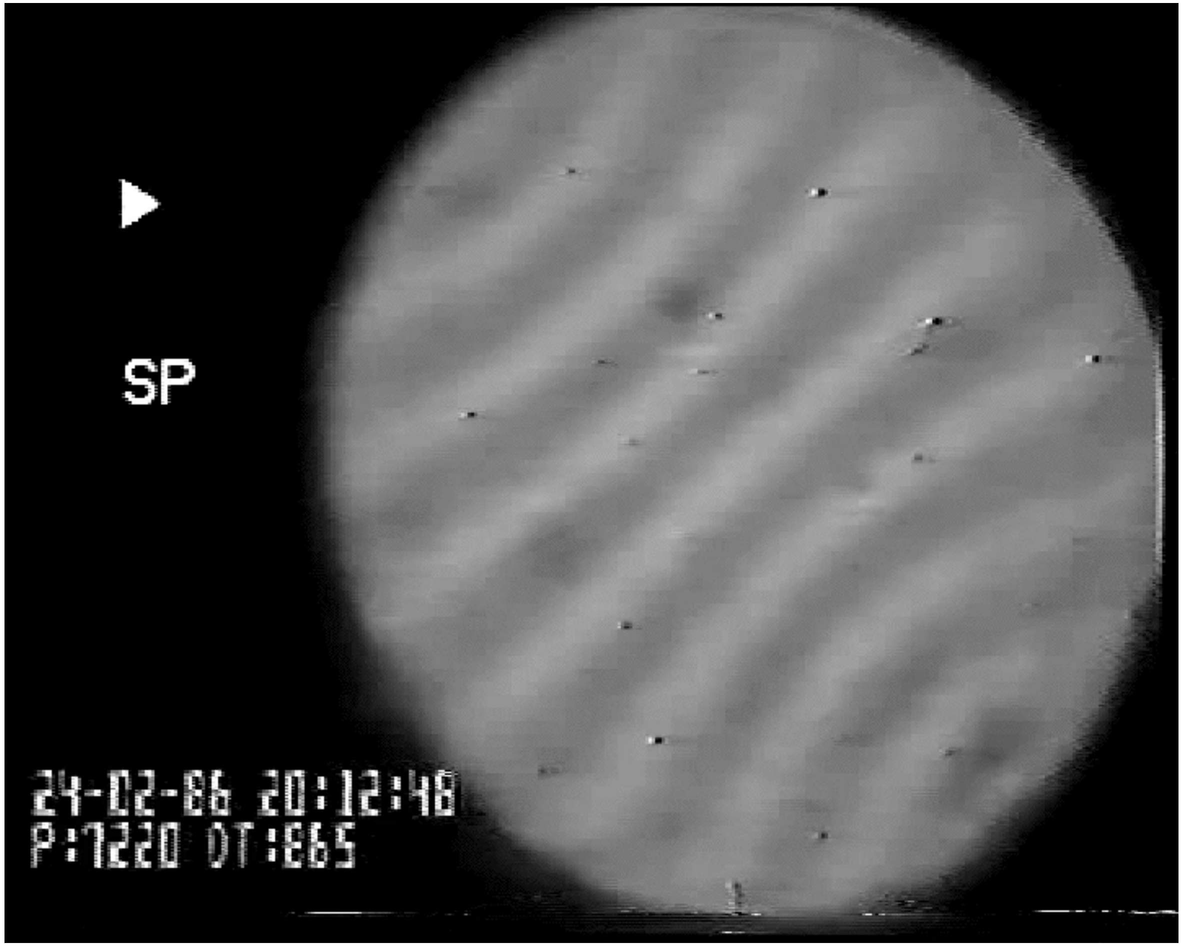
Phase dynamics in convective structures

Permanent dynamics

Convection in gaz : same Prandtl number than liquid He : Pr=0.7

Increase of pressure → visualization

What phenomenon
prevents steadiness ?



A. Pocheau, V. Croquette, P. Le Gal, PRL55,1094,1985

Mean flow mode

Galilean invariance

Galilean invariance : $(\mathbf{r}, t, \mathbf{V}) \longrightarrow (\mathbf{r} + \mathbf{V_0}\, t, t, \mathbf{V} + \mathbf{V_0})$

uniform flow **U** : neutral mode : $\partial_{x_i}\mathbf{U} = \mathbf{0} \Rightarrow \frac{\partial \mathbf{U}}{\partial t} = \mathbf{0}$

long wavelength flow : slow mode : $\frac{\partial \mathbf{U}}{\partial t} \approx \mathbf{0}$

nature of **U** : **mean flow**

2 slow modes : (φ, **U**) → co-dimension 2 dynamics → oscillations permitted

top and bottom rigid boundaries → damped flow mode → co-dimension 1 dynamics with coupling phase/flow

P.Coullet, S.Fauve,PRL55,2857,1985

Mean flow mode

Physical origin

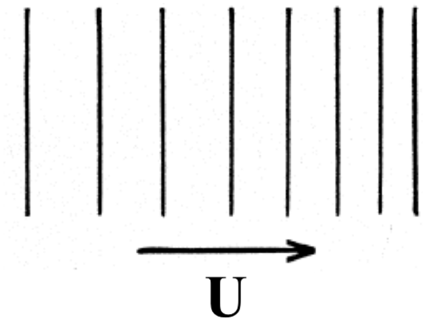
Hydrodynamics

$(\mathbf{V}.\nabla)\mathbf{V}$ → quadratic interaction

$V = A\exp(ikx) + cc$ → $\exp(ikx) \times \exp(-ikx) \equiv \exp(0x)$: spatial beating

→ $$V\frac{\partial}{\partial x}V = \frac{1}{2}\frac{\partial |A|^2}{\partial x} + harmonics$$

→ requires a breaking of
the parity symmetry



However, a pressure gradient compensates divergent flows

→ the resulting mean flow U : is induced by the rotational part of mean flow sources

corresponds to a large scale vertical vorticity

Other systems :

liquid crystals : permeation process
Taylor-Couette flows
waves
zonal flows
river meandering

E.D.Siggia, A.Zippelius,PRL47,835,1981

Phase advection-diffusion dynamics

$$\frac{\partial \varphi}{\partial t} + (\mathbf{U}.\nabla)\varphi = D_{//}\frac{\partial^2 \varphi}{\partial x^2} + D_{\perp}\frac{\partial^2 \varphi}{\partial y^2} + h.o.t.$$

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}.\nabla)\mathbf{U} = \underbrace{\nu \triangle \mathbf{U}}_{\text{viscous dissipation}} \underbrace{- \mu \mathbf{k}\nabla(\mathbf{k}|A|^2)}_{\text{mean flow sources}} + \underbrace{\nabla \mathit{\Pi}}_{\text{large scale pressure gradient}} + h.o.t.$$

d : distance between top and bottom boundaries

L : modulation wavlength

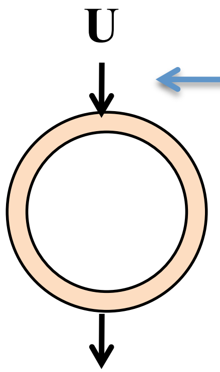
$$\left.\begin{array}{l}\tau_U \approx d^2/\nu \\ \tau_\varphi \approx L^2/D\end{array}\right\}$$

d << L : $\tau_U \ll \tau_\varphi$ → φ : single slow mode
with mean flow coupling

Phase advection-diffusion dynamics

Phase compression

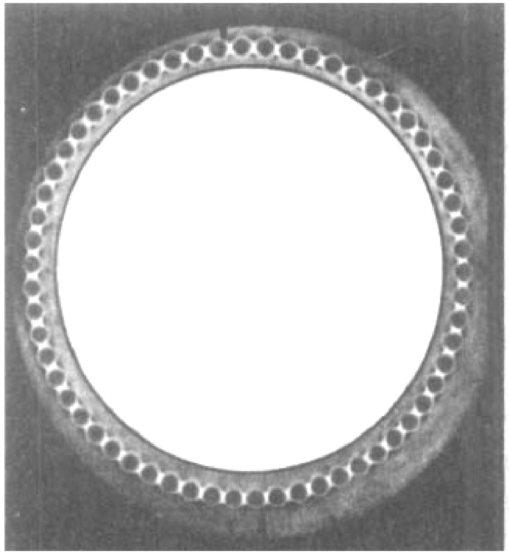
Imposed U



external mean flow generated by a thermosyphon

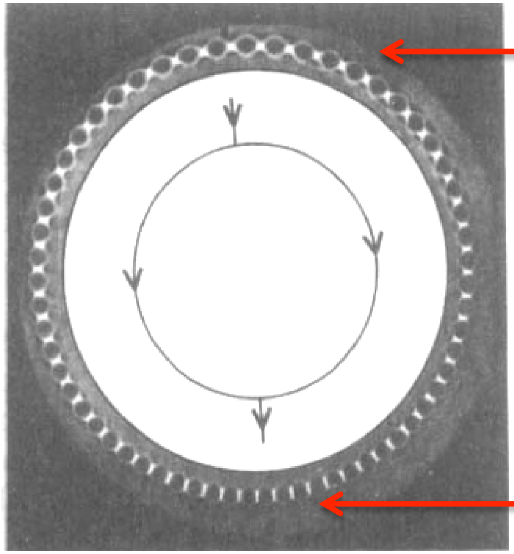
annular fluid domain

$\mathbf{U} = \mathbf{0}$



uniform rolls

$\mathbf{U} \neq \mathbf{0}$



wavenumber gradient

small wavenumbers

large wavenumbers

roll chains crossed by a mean flow
no global advection

phase compression

A.Pocheau, V.Croquette, P.LeGal, C.Poitou,EPL3,915,1987

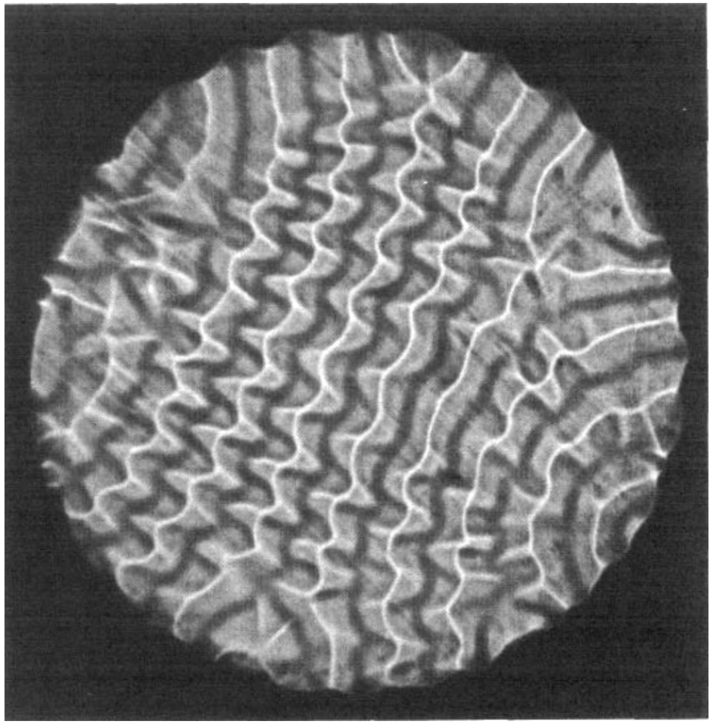
Phase advection-diffusion dynamics

Oscillatory instability

short scale instability : L ≈ λ ≈ d

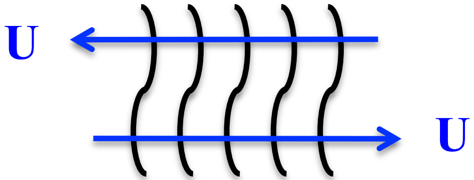
$$\left.\begin{aligned}\tau_U &\approx d^2/\nu \\ \tau_\varphi &\approx L^2/D\end{aligned}\right\}$$

d ≈ L : $\tau_U \approx \tau_\varphi$



co-dimension 2 dynamics

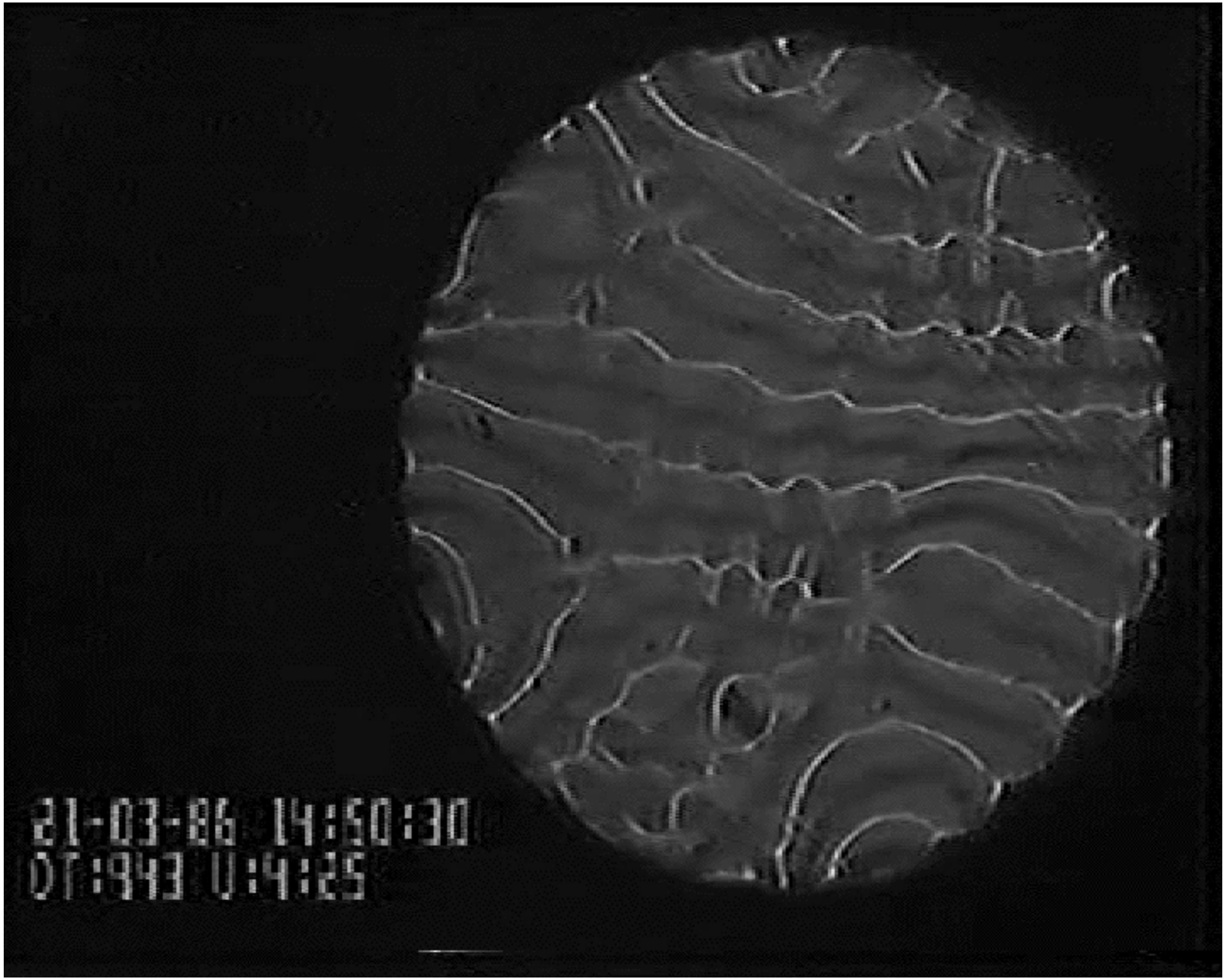
→ oscillations



V.Croquette, P.LeGal, A.Pocheau, PhysicaScripta13,135,1986
M. Cross, A. Newell, PhysicaD10,299,1984
F.H. Busse, R.M. Clever, J.FluidMech.102,61,1981
F.H.Busse, Rep.Prog.Phys.41,1929,1978

Phase advection-diffusion dynamics

Oscillatory instability



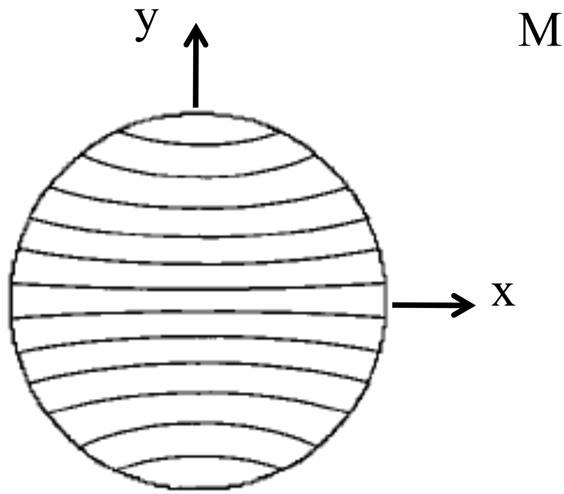
Phase advection-diffusion dynamics

Permanent dynamics at onset

Mechanism :

curvature → mean flows → phase compression → local instability → time-dependence

Phase dynamics equation
Mean flow equations
} algebraic expansion of phase distortion



$$\varphi(x,y)=k_0(1+\Delta)y\left[1-a\frac{x^2}{R^2}+b\frac{y^2}{R^2}+c\frac{y^4}{R^4}+d\frac{x^2y^2}{R^4}\right]$$

Steady solution at dominant order : flow field and phase distortion

→ (a, b, c, d) function of ε

→ wavenumber band

A.Pocheau, J.PhysFrance49,1127,1988

Phase advection-diffusion dynamics

Permanent dynamics at onset

Mechanism :

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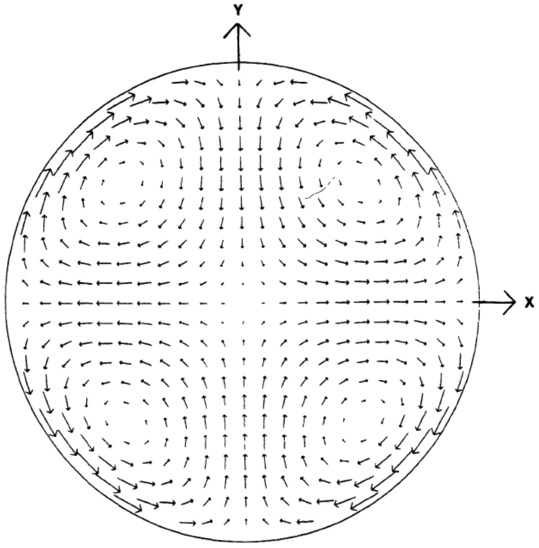
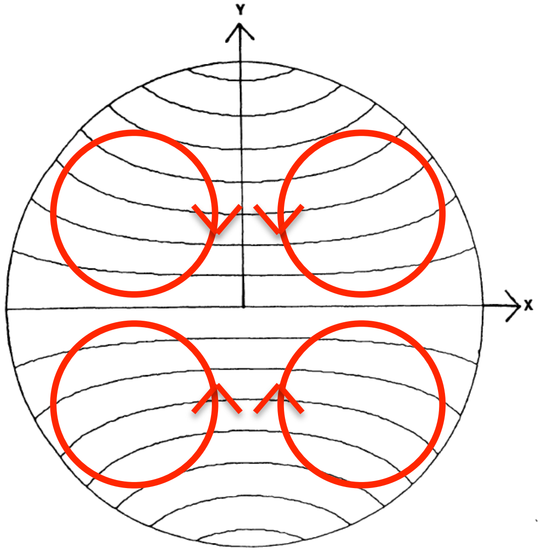
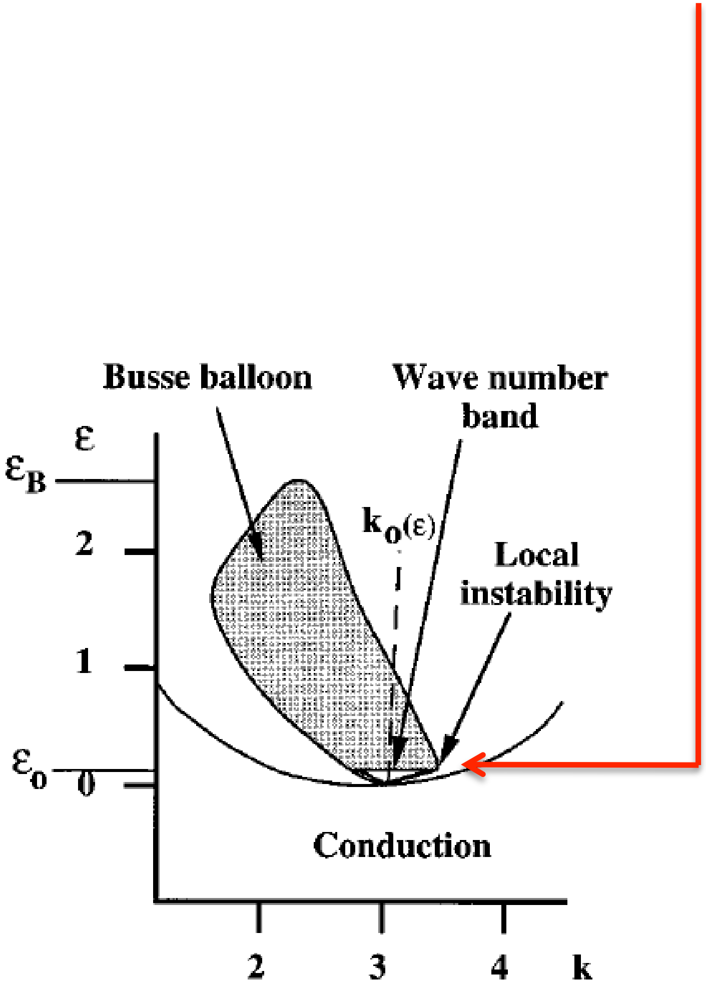


Mean flow



Mean flow



Convective structure that is self-stressed
by its own mean flows

A.Pocheau, J.PhysFrance49,1127,1988

Phase advection-diffusion dynamics

Controlling phase dynamics from mean flows

Mean flows determined by :

- their source : phase field
- their boundary conditions

→ playing on mean flow b.c. to monitor the mean flow implications on a given convective structure

Closed containers : same b.c. for the convective structure and the mean flows

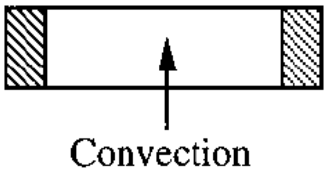
→ B.C. are set by a solid :

at a point of the cell depth

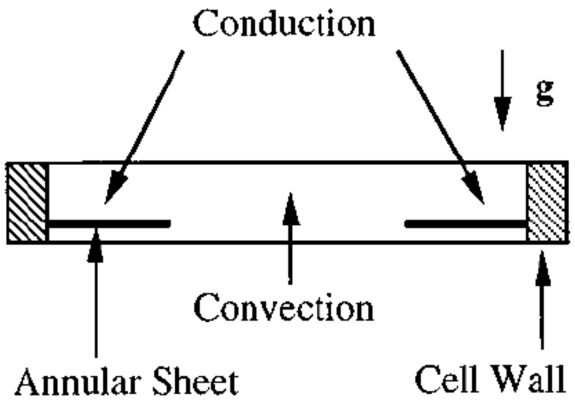
in the whole cell depth

Separate b.c. :

→ annular sheet in the cell depth



Closed container
for mean flows



Open container
for mean flows

F.Daviaud, A.Pocheau, EPL9,675,1988
A.Pocheau, F.Daviaud, PRE55,353,1997

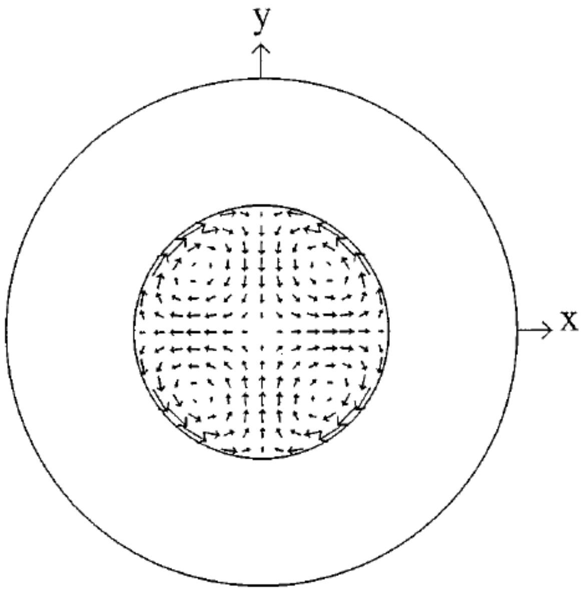
Phase advection-diffusion dynamics

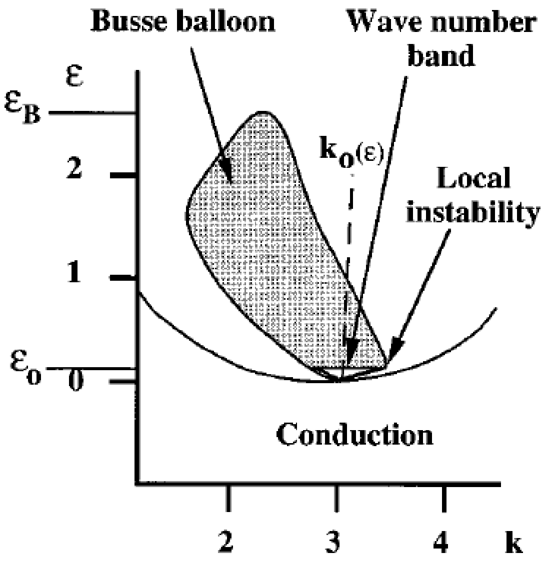
Controlling phase dynamics from mean flows

Closed container

↓

phase turbulence
close to onset





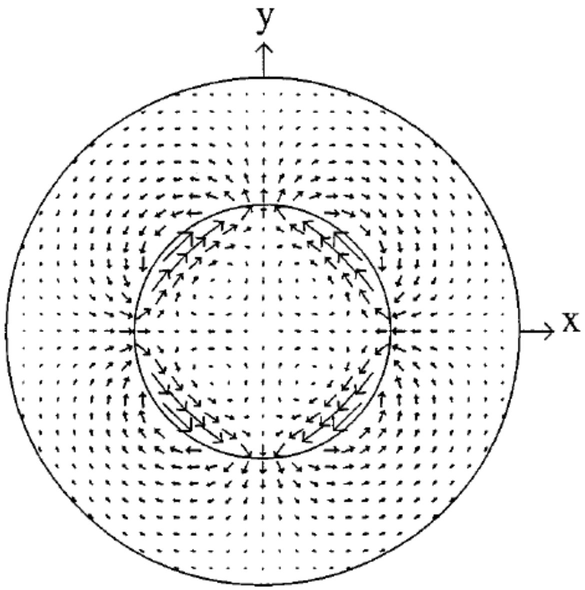
Open container

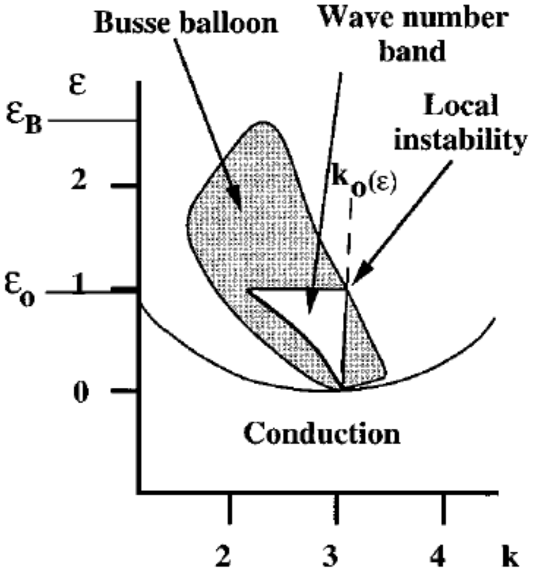
↓

phase turbulence
far to onset

↓

phase turbulence
inhibited





F.Daviaud, A.Pocheau, EPL9,675,1988
A.Pocheau, F.Daviaud, PRE55,353,1997

Conclusion

Symetries → slow modes → master modes for pattern dynamics

Phase mode : generic mode for patterns involving translational invariance

Mean flow mode : generic mode for patterns involving galilean invariance

usually coupled
- → phase instabilities
- → permanent phase dynamics
  - → can be enhanced or inhibited from b.c.

Phase mode : easily perturbed by other fields → distortion or dynamics

Universal analysis and concepts

... to adapt to each physical context